Control Volume Derivation

- How to convert our relationships for a closed system (control mass) to an open system (control volume)
- For mass conservation, our control mass “law” was
  \[ \frac{dm_{sys}}{dt} = 0 \]
- In integral form (integrating over control mass),
  \[ \frac{d}{dt} \int_{CM} \rho dV = 0 \]

Control Mass/Control Volume

- Consider general control mass and control volumes that are moving in time; coincide at time \( t \)
- Want to see how to convert CM law to CV law
  \[ \frac{d}{dt} \int_{CM} \rho(x, t) dV = 0 \]
- Must integrate 3-d integral over time dependent domain

\( v_{rel} = \) relative velocity of material crossing CV’s CS
\( dA = \) differential area element on CV’s CS

Control Surfaces

\( CM(t) = CV(t) \)
Time Derivative of 3-d Integral

• Start with standard limit value definition of derivative

\[
\frac{d}{dt} \int_{CV(t)} \rho(x, t) \, dV = \lim_{\Delta t \to 0} \left[ \frac{\int_{CM(t+\Delta t)} \rho(x, t+\Delta t) \, dV - \int_{CM(t)} \rho(x, t) \, dV}{\Delta t} \right]
\]

but \[ \int_{CM(t+\Delta t)} \rho(x, t+\Delta t) \, dV = \int_{CV(t+\Delta t)} \rho(x, t+\Delta t) \, dV + \int_{\text{Shaded Region}} \rho(x, t+\Delta t) \, dV \]

\[
\frac{d}{dt} \int_{CM(t)} \rho(x, t) \, dV = \lim_{\Delta t \to 0} \left[ \frac{\int_{CV(t+\Delta t)} \rho(x, t+\Delta t) \, dV - \int_{CV(t)} \rho(x, t) \, dV}{\Delta t} + \frac{1}{\Delta t} \int_{\text{Shaded Region}} \rho(x, t+\Delta t) \, dV \right]
\]

Time Derivative of 3-d Integral (con't)

• But \[ \] term is definition of derivative

\[
\lim_{\Delta t \to 0} \left[ \frac{\int_{CV(t+\Delta t)} \rho(x, t+\Delta t) \, dV - \int_{CV(t)} \rho(x, t) \, dV}{\Delta t} \right] = \frac{d}{dt} \int_{CV(t)} \rho \, dV
\]

• Use \( dV = \vec{v}_{\text{rel}} \Delta t \cdot \vec{n} \, dA \) with \( \vec{n} \) a vector normal to dA and pointed outward

• Shaded region term becomes

\[
\lim_{\Delta t \to 0} \left[ \frac{1}{\Delta t} \int_{\text{Shaded Region}} \rho(x, t+\Delta t) \, dV \right] = \lim_{\Delta t \to 0} \left[ \frac{\Delta t}{\Delta t} \int_{\text{Shaded Region}} \rho(\vec{v}_{\text{rel}} \cdot \vec{n}) \, dA \right] = \int_{CS(t)} \rho(\vec{v}_{\text{rel}} \cdot \vec{n}) \, dA
\]
Control Volume Form of Mass Conservation

• From previous two equations, we have
  \[
  \frac{d}{dt} \int_{CM(t)} \rho dV = \frac{d}{dt} \int_{CV(t)} \rho dV + \int_{CS(t)} \rho (\vec{v}_{rel} \cdot \vec{n}) dA
  \]

• Applying mass conservation (LHS=0)
  \[
  0 = \frac{d}{dt} \int_{CV(t)} \rho dV + \int_{CS(t)} \rho (\vec{v}_{rel} \cdot \vec{n}) dA
  \]

  Production rate of mass, \( P_{mass} \)

  Time rate of change of mass inside CV

  Net outward mass flow rate crossing CS

  Don’t have to know mass distribution in CV

Simplifications

• Uniform flow (at CS)
  \[
  \int_{CS(t)} \rho (\vec{v}_{rel} \cdot \vec{n}) dA = \sum_{inlets} \rho v_{rel} A - \sum_{outlets} \rho v_{rel} A
  \]
  i.e., \( \dot{m} = \rho v_{rel} A \)

  + Working in frame of reference where CS not moving
    \[
    = \sum_{outlets} \rho v A - \sum_{inlets} \rho v A = \sum_{outlets} m - \sum_{inlets} \dot{m}
    \]

• Steady-State
  \[
  \frac{d}{dt} \int_{CV(t)} \rho dV = 0 \Rightarrow \sum_{outlets} \dot{m} = \sum_{inlets} \dot{m}
  \]
Simplifications (con’t)

- Transient, integrate over fixed time

\[
\int_{t_1}^{t_2} \left[ \frac{d}{dt} \int_{CV(t)} \rho dV \right] dt = \int_{t_1}^{t_2} \left[ \sum_{\text{inlets}} \dot{m} - \sum_{\text{outlets}} \dot{m} \right] dt
\]

\[
\int_{CV(t_2)}^{CV(t_1)} \rho dV - \int_{CV(t_1)}^{CV(t_2)} \rho dV = \int_{t_1}^{t_2} \left[ \sum_{\text{inlets}} \frac{dm}{dt} - \sum_{\text{outlets}} \frac{dm}{dt} \right] dt
\]

\[
m_{CV,2} - m_{CV,1} = \sum_{\text{inlets}} \Delta m_{12} - \sum_{\text{outlets}} \Delta m_{12}
\]

Change of mass in CV between \( t_1 \) and \( t_2 \)

Net amount of mass entering CV between \( t_1 \) and \( t_2 \)

In-Class Problems

1. Nitrogen at 200 kPa and 25°C flows through a 35 mm diam. pipe at 20 m/s. Find the mass flow rate of nitrogen through the pipe.

2. Liquid water enters the square duct shown with an average velocity 10 m/s. Determine the average velocity and mass flow rate at the exit.

3. Air flows in a circular pipe with a velocity of 20 m/s. Around the pipe, in an annulus, is a 2nd flow of air, with a velocity of 40 m/s. Both flows exhaust into a 15 cm diam. pipe. If the flow at \( e \) is uniform, determine the flow velocity at \( e \). Assume the air density is constant.
Differential Form of Mass Conservation
For Quasi-1D, Steady Flow

• Assume flow velocity is 1-D (only variation in x) in non-constant area, differential volume
  \[ \dot{m} = \text{constant} = \rho v A \]
  \[ 0 = d(\rho v A) \]
  \[ 0 = v A \dot{\rho} + \rho A \dot{v} + \rho v dA \]

\[ 0 = \frac{\dot{\rho}}{\rho} v + \frac{\dot{v}}{v} \frac{dA}{A} = \frac{\dot{\rho}}{\rho} + \frac{1}{2} \frac{d(v^2)}{v^2} + \frac{dA}{A} \]

• Compare to continuity
  \[ 0 = \frac{\dot{\rho}}{\rho} v + \frac{d}{dx} \left( \frac{d(\rho v_x)}{dx} + \frac{d(\rho v_y)}{dy} + \frac{d(\rho v_z)}{dz} \right) = \frac{d(\rho v)}{dx} \Rightarrow \text{strictly, } \frac{dA}{dx} = 0 \]

Reynolds Transport Theorem

• Provides general form for converting from CM to CV conservation laws

• For given extensive property B, with intensive version \( \beta \) (something per mass), that follows a “conservation” law can show

\[ \frac{dB}{dt}_{\text{CM}} = \frac{d}{dt} \int_{\text{CV}} \rho \beta dV + \int_{\text{CS}} \rho \beta (\vec{v}_{\text{rel}} \cdot \vec{n}) dA \]

Replace with appropriate Control Mass Conservation Law

• Will also lead to a PICO relationship
  \[ \text{Production} + \text{Input} = \text{Change (in time)} + \text{Output} \]
Reynolds Transport Theorem: Example

- Example, linear momentum
  \[ \vec{B} = m\vec{v}, \quad \vec{\beta} = \vec{\nu} \]

- RTT then gives
  \[ \frac{d(m\vec{v})}{dt} \bigg|_{CM} = \frac{d}{dt} \int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v} (\vec{v}_{rel} \cdot \vec{n}) dA \]

- Use Newton's Law
  - e.g., gravity
  - e.g., pressure (\(p\text{d}A\)), shear stress

\[
\frac{d(m\vec{v})}{dt} \bigg|_{CM} = \sum \vec{F}_{on\, CV} = \sum \vec{F}_{body\, on\, CV} + \sum \vec{F}_{surface\, on\, CS} (\equiv \dot{P}_{momentum}) \\
\dot{P}_{momentum} = \sum \vec{F}_{on\, CV} = \frac{d}{dt} \int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v} (\vec{v}_{rel} \cdot \vec{n}) dA
\]

Momentum Conservation: 1-D Flow

- For steady inviscid flow, no body forces

\[
\sum \vec{F}_{x\, body\, on\, CV} + \sum \vec{F}_{x\, stresses\, from\, solid\, CS} - \int_{CS} \rho \vec{n} \cdot dA = \int_{CV} \frac{d}{dt} \left( \rho \vec{v} dV \right) + \int_{CS} \rho \vec{v} (\vec{v}_{rel} \cdot \vec{n}) dA \\
\vec{F}_{x\, shear\, stress\, on\, CS} = \int_{CS} \rho \vec{n} dA = \sum \vec{m} \vec{v} - \sum \vec{m} \vec{v} \\
\vec{F}_{x} + p_{1}A_{1} - p_{2}A_{2} + p_{amb}(A_{2} - A_{1}) = \vec{m}v_{2} - \vec{m}v_{1} = \rho_{2}v_{2}^{2}A_{2} - \rho_{1}v_{1}^{2}A_{1} \\
\vec{F}_{x} + (p_{1} - p_{amb})A_{1} - (p_{2} - p_{amb})A_{2} = \vec{m}v_{2} - \vec{m}v_{1} = \rho_{2}v_{2}^{2}A_{2} - \rho_{1}v_{1}^{2}A_{1}
\]
Momentum Conservation: 1-D Flow

• Differential form

\[ -\tau_x L_p dx + p A - \left( p + \frac{dp}{2} \right) \frac{dA}{p} = \bar{m} (v + dv) - \bar{m} v \]

\[ -\left( p + \frac{dp}{2} \right) \frac{dA}{A} = \bar{m} (v + dv) - \bar{m} v \]

For steady, no body forces and shear stress defined to be in -x direction.

Flow Rates and Fluxes

• Flow rate of property that is carried by the mass crossing the control surface (in some direction) is

\[ B = \int \rho \beta (\bar{v}_{rel} \cdot \bar{n}) dA \]

e.g., \( \bar{m} = \rho v A \) (kg/s).

\[ \bar{m} v = \rho v^2 A \] (N) for 1-D flow with stationary CV

• The flux of that same property is given by

\[ \bar{B}^* = \rho \beta (\bar{v}_{rel} \cdot \bar{n}) \]

e.g., \( \rho v \) (mass flux, kg/s/m²),

\[ \rho v^2 \] (momentum flux, N/m²) for nonmoving CS