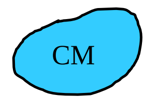
Control Volume Derivation

- How to convert our relationships for a closed system (control mass) to an open system (control volume)
- For mass conservation, our control mass "law" was

$$\frac{dm_{sys}}{dt} = 0$$



- In integral form (integrating over control mass),

$$\frac{d}{dt}\int_{CM} \rho dV = 0$$

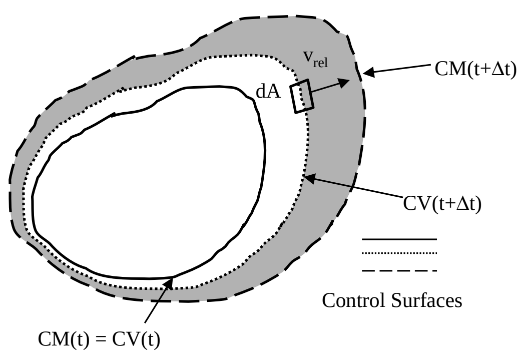
Control Mass/Control Volume

- Consider general control mass and control volumes that are moving in time; coincide at time t
- Want to see how to convert CM law to CV law

$$\frac{d}{dt}\int_{CM} \rho(\vec{x}, t) dV = 0$$

- Must integrate 3-d integral over time dependent domain

v_{rel}=relative velocity of material crossing CV's CS
dA =differential area element on CV's CS



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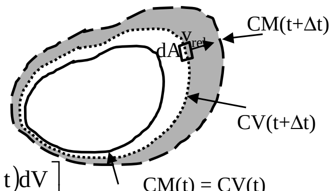
Time Derivative of 3-d Integral

- Start with standard limit value definition of derivative



$$\frac{d}{dt}\int_{CM(t)}\rho(\vec{x},t)dV = \lim_{\Delta t\to 0}\left[\frac{\int_{CM(t+\Delta t)}\rho(\vec{x},t+\Delta t)dV - \int_{CM(t)}\rho(\vec{x},t)dV}{\Delta t}\right]$$

but $$\int_{CM(t+\Delta t)}\rho(\vec{x},t+\Delta t)dV = \int_{CV(t+\Delta t)}\rho(\vec{x},t+\Delta t)dV + \int_{\substack{\text{Shaded}\\ \text{Region}}}\rho(\vec{x},t+\Delta t)dV$$

$$\frac{d}{dt}\int_{CM(t)}\rho(\vec{x},t)dV = \lim_{\Delta t\to 0}\left\{\left[\frac{\int_{CV(t+\Delta t)}\rho(\vec{x},t+\Delta t)dV - \int_{CV(t)=CM(t)}\rho(\vec{x},t)dV}{\Delta t}\right] + \frac{1}{\Delta t}\int_{\substack{\text{Shaded}\\ \text{Region}}}\rho(\vec{x},t+\Delta t)dV\right\}$$

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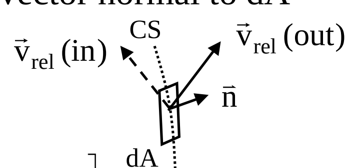
Time Derivative of 3-d Integral (con't)

- But [ ] term is definition of derivative

$$\lim_{\Delta t\to 0}\left[\frac{\int_{CV(t+\Delta t)}\rho(\vec{x},t+\Delta t)dV - \int_{CV(t)}\rho(\vec{x},t)dV}{\Delta t}\right] = \frac{d}{dt}\int_{CV(t)}\rho dV$$

- Use $dV = \vec{v}_{rel}\Delta t\cdot\vec{n}\,dA$ with $\vec{n}$ a vector normal to dA and pointed outward



- Shaded region term becomes

$$\lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}\int_{\substack{\text{Shaded}\\ \text{Region}}}\rho(\vec{x},t+\Delta t)dV\right] = \lim_{\Delta t\to 0}\left[\frac{\Delta t}{\Delta t}\int_{\substack{\text{CS of}\\ \text{CV at t}}}\rho(\vec{v}_{rel}\cdot\vec{n})dA\right] = \int_{CS(t)}\rho(\vec{v}_{rel}\cdot\vec{n})dA$$

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Control Volume Form of Mass Conservation

- From previous two equations, we have

$$\frac{d}{dt}\int_{CM(t)}\rho dV = \frac{d}{dt}\int_{CV(t)}\rho dV + \int_{CS(t)}\rho(\vec{v}_{rel}\cdot\vec{n})dA$$

- Applying mass conservation (LHS=0)

$$\boxed{0 = \underbrace{\frac{d}{dt}\int_{CV(t)}\rho dV}_{\text{Time rate of change of mass inside CV}} + \underbrace{\int_{CS(t)}\rho(\vec{v}_{rel}\cdot\vec{n})dA}_{\text{Net outward mass flow rate crossing CS}}}$$

Production rate of mass, $\dot{P}_{mass}$

$$\dot{P}_{mass} = 0 = \frac{dm_{CV}}{dt} + \sum_{outlets}\dot{m} - \sum_{inlets}\dot{m}$$

Don't have to know mass <u>distribution</u> in CV

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Simplifications

- **Uniform flow (at CS)**

$$\int_{CS(t)}\rho(\vec{v}_{rel}\cdot\vec{n})dA = \sum_{outlets}\rho v_{rel}A - \sum_{inlets}\rho v_{rel}A$$

$$\text{i.e., } \dot{m} = \rho v_{rel}A$$

$+$ Working in frame of reference where CS <u>not moving</u>

$$= \sum_{outlets}\rho vA - \sum_{inlets}\rho vA =$$

- **Steady-State**

$$\frac{d}{dt}\int_{CV(t)}\rho dV = 0 \Rightarrow \sum_{outlets}\dot{m} = \sum_{inlets}\dot{m}$$

Simplifications (con't)

- **Transient, integrate over fixed time**

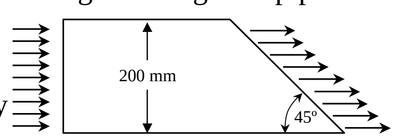
$$\int_{t_1}^{t_2}\left[\frac{d}{dt}\int_{CV(t)}\rho dV\right]dt = \int_{t_1}^{t_2}\left[\sum_{inlets}\dot{m} - \sum_{outlets}\dot{m}\right]dt$$

$$\int_{CV(t_2)}\rho dV - \int_{CV(t_1)}\rho dV = \int_{t_1}^{t_2}\left[\sum_{inlets}\frac{dm}{dt} - \sum_{outlets}\frac{dm}{dt}\right]dt$$

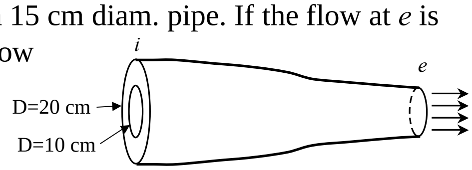
$$\underbrace{m_{CV,2} - m_{CV,1}}_{\substack{\text{Change of mass in CV} \\ \text{between } t_1 \text{ and } t_2}} = \underbrace{\sum_{inlets}\Delta m_{12} - \sum_{outlets}\Delta m_{12}}_{\substack{\text{Net amount of mass entering} \\ \text{CV between } t_1 \text{ and } t_2}}$$

In-Class Problems

1. Nitrogen at 200 kPa and 25^0C flows through a 35 mm diam. pipe at 20 m/s. Find the mass flow rate of nitrogen through the pipe.
2. Liquid water enters the square duct shown with an average velocity 10 m/s. Determine the average velocity and mass flow rate at the exit.

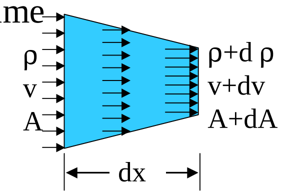


3. Air flows in a circular pipe with a velocity of 20 m/s. Around the pipe, in an annulus, is a 2nd flow of air, with a velocity of 40 m/s. Both flows exhaust into a 15 cm diam. pipe. If the flow at *e* is uniform, determine the flow velocity at *e*. Assume the air density is constant.



Differential Form of Mass Conservation For Quasi-1D, Steady Flow

- Assume flow velocity is 1-D (only variation in x) in non-constant area, differential volume



$$\dot{m} = \text{constant} = \rho v A$$
$$0 = d(\rho v A)$$
$$0 = vAd\rho + \rho A dv + \rho v dA$$
$$\boxed{0 = \frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = \frac{d\rho}{\rho} + \frac{1}{2}\frac{d\left(v^2\right)}{v^2} + \frac{dA}{A}}$$

- Compare to continuity

$$0 = \underbrace{\cancel{\frac{d\rho}{dt}}}_{0} + \left[\frac{d(\rho v_x)}{dx} + \underbrace{\cancel{\frac{d\left(\rho v_y\right)}{dy}}}_{0} + \underbrace{\cancel{\frac{d(\rho v_z)}{dz}}}_{0}\right] = \frac{d(\rho v)}{dx} \Rightarrow \left(\text{strictly, } \frac{dA}{dx} = 0\right)$$

Reynolds Transport Theorem

- Provides general form for converting from CM to CV conservation laws
- For given extensive property B, with intensive version β (something per mass), that follows a "conservation" law can show

$$\boxed{\left.\frac{dB}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV} \rho\beta dV + \int_{CS} \rho\beta\left(\vec{v}_{rel} \cdot \vec{n}\right)dA}$$

Replace with appropriate Control Mass Conservation Law

- Will also lead to a **PICO** relationship

Production + **I**nput = **C**hange (in time) + **O**utput

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Reynolds Transport Theorem: Example

- Example, **linear momentum**

$$\vec{B} = m\vec{v}, \quad \vec{\beta} = \vec{v}$$

- **RTT** then gives

$$\left.\frac{d(m\vec{v})}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV}\rho\vec{v}\,dV + \int_{CS}\rho\vec{v}\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

- Use **Newton's Law**

e.g., gravity

e.g., pressure ($\int p\,dA$), shear stress

$$\left.\frac{d(m\vec{v})}{dt}\right|_{CM} = \sum\vec{F}_{on\,CV} = \sum\vec{F}_{\substack{body\\ on\,CV}} + \sum\vec{F}_{\substack{surface\\ on\,CS}}\left(= \dot{P}_{momentum}\right)$$

$$\dot{P}_{momentum} = \sum\vec{F}_{on\,CV} = \frac{d}{dt}\int_{CV}\rho\vec{v}\,dV + \int_{CS}\rho\vec{v}\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

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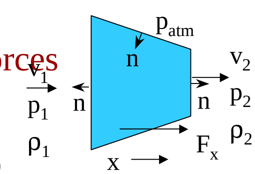
Momentum Conservation: 1-D Flow

- For steady inviscid flow, no body forces



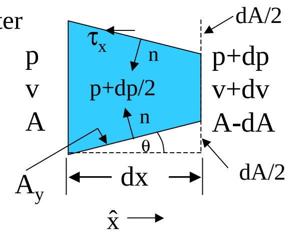
$$\sum\underbrace{\cancel{\vec{F}_{x\,\substack{body\\ on\,CV}}}}_{0} + \sum\vec{F}_{x\,\substack{stresses\,from\\ solid\,CS}} - \int_{CS} p\cdot\vec{n}\,dA = \underbrace{\cancel{\frac{d}{dt}\int_{CV}\rho\vec{v}\,dV}}_{0} + \int_{CS}\rho\vec{v}\left(\vec{v}_{rel}\cdot\vec{n}\right)dA$$

$$\vec{F}_{x\,\substack{shear\,stress\\ on\,CS}} - \int_{CS} p\cdot\vec{n}\,dA = \sum_{outlets}\dot{m}\vec{v} - \sum_{inlets}\dot{m}\vec{v}$$

$$\vec{F}_x + p_1A_1 - p_2A_2 + p_{amb}\left(A_2 - A_1\right) = \dot{m}v_2 - \dot{m}v_1 = \rho_2 v_2^2 A_2 - \rho_1 v_1^2 A_1$$

$$\boxed{\vec{F}_x + \left(p_1 - p_{amb}\right)A_1 - \left(p_2 - p_{amb}\right)A_2 = \dot{m}v_2 - \dot{m}v_1 = \rho_2 v_2^2 A_2 - \rho_1 v_1^2 A_1}$$

Momentum Conservation: 1-D Flow



- Differential form

$$-\tau_x L_p dx + pA - (p+dp)(A-dA)$$
$$-\left(p+\frac{dp}{2}\right)dA = \dot{m}(v+dv) - \dot{m}v$$
$$-\tau_x L_p dx - Adp = \dot{m}dv = \rho vAdv$$

$$\boxed{\frac{\tau_x}{p}\frac{L_p}{A}dx + \frac{dp}{p} + \frac{\rho v^2}{2p}\frac{dv^2}{v^2} = 0}$$

- Term comes from

$$F_{x,sides} = \left(p+\frac{dp}{2}\right)(\vec{n}\cdot\hat{x})A_y$$
$$= \left(p+\frac{dp}{2}\right)\sin\theta\frac{dA}{\sin\theta}$$

– For steady, no body forces and shear stress defined to be in -x direction

Flow Rates and Fluxes

- Flow rate of property that is carried by the mass crossing the control surface (in some direction) is

$$B = \int \rho\beta(\vec{v}_{rel}\cdot\vec{n})dA$$

e.g., $\dot{m} = \rho vA$ (kg/s),

$\dot{m}v = \rho v^2 A$ (N) for 1–D flow with stationary CV

- The flux of that same property is given by

$$\dot{B}'' = \rho\beta(\vec{v}_{rel}\cdot\vec{n})$$

e.g., ρv (mass flux, kg/s/m^2),

ρv^2 (momentum flux, N/m^2) for nonmoving CS