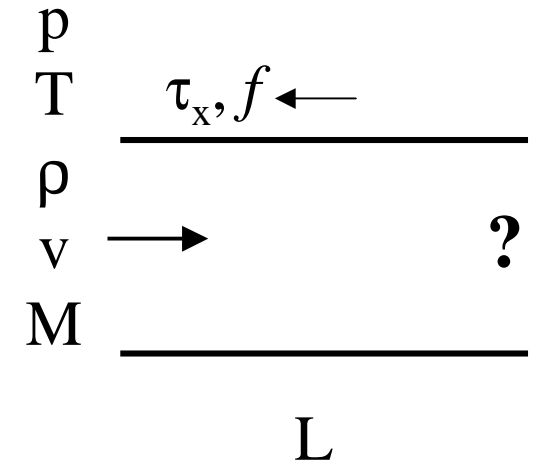


Fanno Flow - Thermodynamics

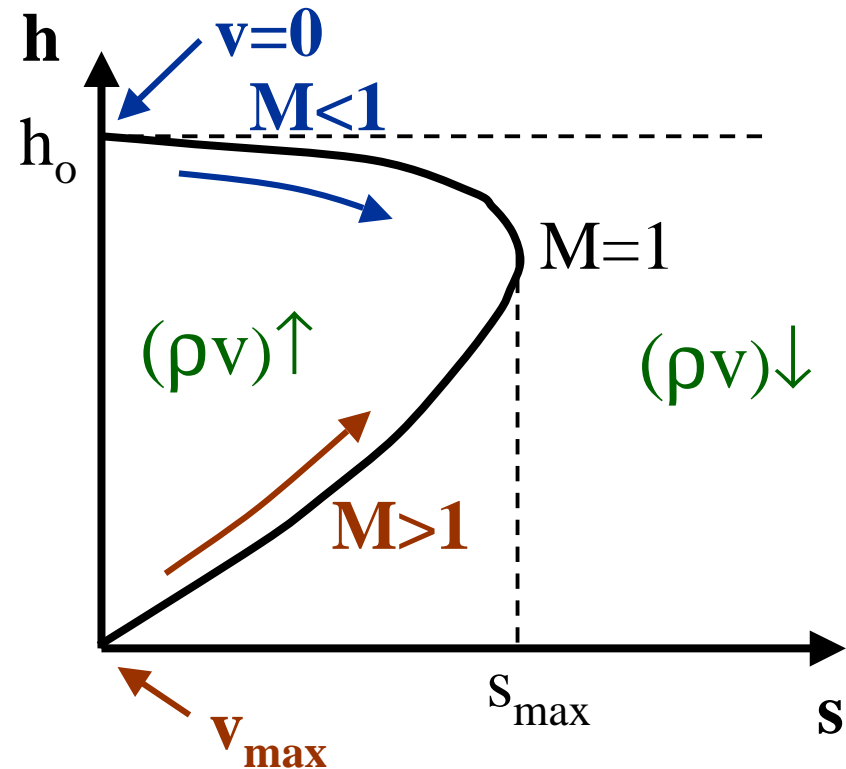
- Steady, 1-d, constant area, adiabatic flow with no external work but *with friction*
- Conserved quantities
 - since adiabatic, no work: $h_o = \text{constant}$
 - since $A = \text{const}$: mass flux = $\rho v = \text{constant} \equiv G$
 - combining: $h_o = h + G^2 / 2\rho^2 = \text{constant}$
- On h - s diagram, can draw **Fanno Line**
 - line connecting points with same h_o and ρv



As you change h , you change ρ (and v) since G and h_o const.

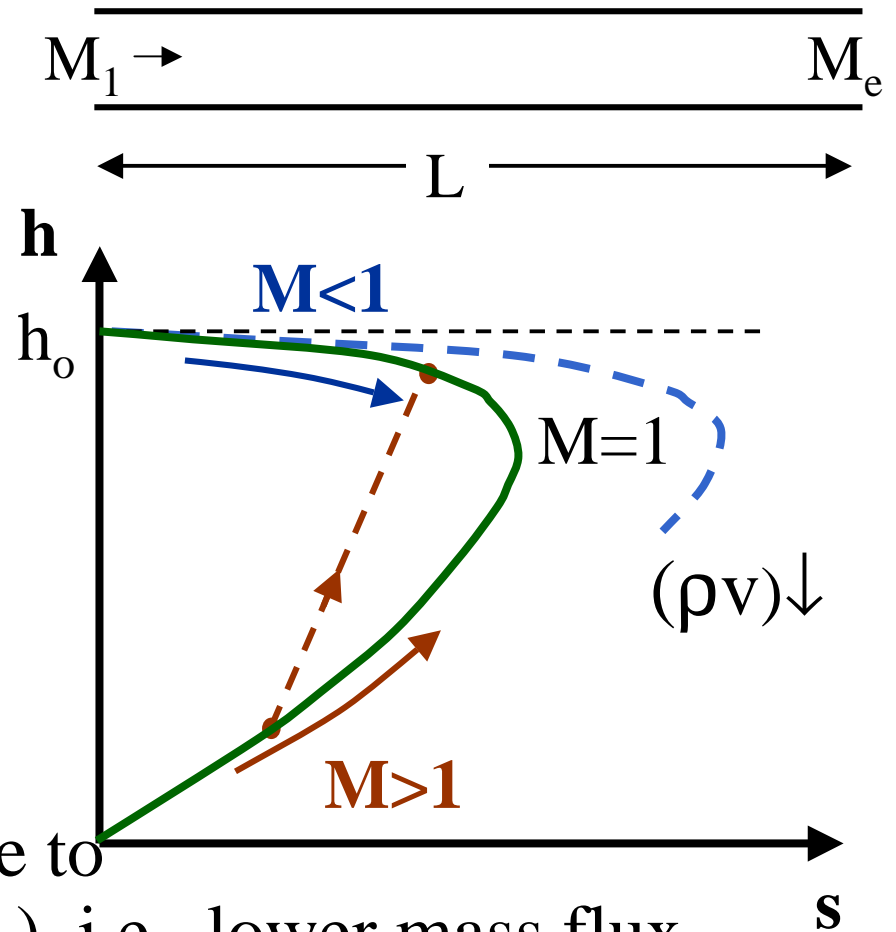
Fanno Line

- Velocity change (due to friction) associated with entropy change
- Friction can only increase entropy
 - can only approach $M=1$
 - friction alone can not allow flow to transition between sub/supersonic
- Two solutions given $(\rho v, h_o, s)$: subsonic & supersonic
 - change mass flux: new Fanno line



Fanno Line - Choking

- Total friction experienced by flow increases with length of “flow”, e.g., duct length, L
- For long enough duct, $M_e = 1$ ($L = L_{\max}$)
- What happens if $L > L_{\max}$
 - flow already “choked”
 - **subsonic flow**: must move to different Fanno line (---), i.e., lower mass flux
 - **supersonic flow**: get a shock (---)



Fanno Line – Mach Equations

- Simplify (X.4-5) for $\delta q = dA = 0$

$$\frac{dM^2}{M^2} = \frac{\gamma M^2 \left(1 + \frac{\gamma-1}{2} M^2 \right) f dx}{1 - M^2 D} \quad (\text{X.6})$$

$$\frac{dT}{T} = \frac{dh}{h} = \frac{-\gamma(\gamma-1)M^4 f dx}{2(1-M^2) D} \quad (\text{X.8})$$

$$\frac{dp}{p} = \frac{-\gamma M^2 \left[1 + (\gamma-1)M^2 \right] f dx}{2(1-M^2) D} \quad (\text{X.7})$$

$$\frac{d\rho}{\rho} = -\frac{dv}{v} = \frac{-\gamma M^2 f dx}{2(1-M^2) D} \quad (\text{X.9})$$

- can write each as only $f(M)$
- p_o loss due to entropy rise

$$\frac{ds}{R} = -\frac{dp_o}{p_o} = \frac{\gamma M^2 f dx}{2 D} \quad (\text{X.10})$$

Property Variations

- Look at signs of previous equations to see how properties changed by friction as we move along flow
 - $(1-M^2)$ term makes $M < 1$ different than $M > 1$

	$M < 1$	$M > 1$
s	↑	↑
p_o	↓	↓
M	↑	↓
h, T	↓	↑
p	↓	↑
ρ	↓	↑
v	↑	↓

- Friction increases s , $\Rightarrow p_o$ drop

- Friction drives $M \rightarrow 1$

- h_o, T_o const: h, T opposite to M

- p, ρ same as T (like isen. flow)

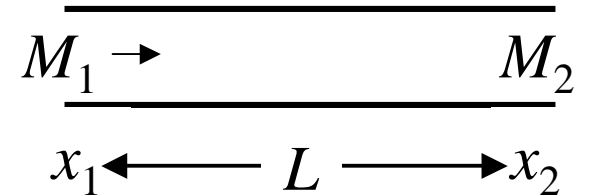
- $\rho v = \text{const}$: v opposite of ρ

Integration

- Need to integrate (X.6-10) to find how properties change along length of flow ($f dx/D$)

– for example, need to integrate

$$\frac{dM^2}{M^2} = \frac{\gamma M^2 \left(1 + \frac{\gamma-1}{2} M^2\right) f dx}{1 - M^2 D}$$



- Separate terms

$$\int_{M_1}^{M_2} \frac{(1 - M^2) dM^2}{\gamma M^4 \left(1 + \frac{\gamma-1}{2} M^2\right)} = \int_{x_1}^{x_2} \frac{f(\text{Re, surface}) dx}{D}$$

f function of Reynolds number (e.g., velocity) and surface roughness

$$\int_{x_1}^{x_2} \frac{f(\text{Re, surface}) dx}{D} \approx \frac{\bar{f}(x_2 - x_1)}{D} = \frac{\bar{f} L}{D}$$

for simplicity, can approximate f by average value

Mach Number Integral

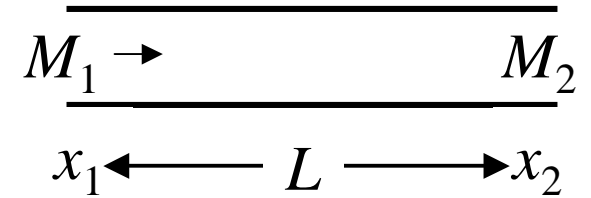
- To perform M integral, redefine variables

$$\begin{aligned}
 \int_{M_1}^{M_2} \frac{(1 - M^2) dM^2}{\gamma M^4 \left(1 + \frac{\gamma - 1}{2} M^2\right)} &= \int_{y_1}^{y_2} \frac{(1 - y) dy}{\gamma y^2 \left(1 + \frac{\gamma - 1}{2} y\right)} \quad \begin{array}{c} \overline{\hspace{10em}} \\ M_1 \rightarrow \hspace{10em} M_2 \\ \overline{\hspace{10em}} \\ x_1 \longleftarrow L \longrightarrow x_2 \end{array} \\
 &= \frac{1}{\gamma} \left[\int_{y_1}^{y_2} \frac{dy}{y^2 \left(1 + \frac{\gamma - 1}{2} y\right)} - \int_{y_1}^{y_2} \frac{dy}{y \left(1 + \frac{\gamma - 1}{2} y\right)} \right] \\
 &= \frac{1}{\gamma} \left[\left\{ \frac{-1}{y} + \frac{\gamma - 1}{2} \ln \frac{1 + \frac{\gamma - 1}{2} y}{y} \right\} - \left\{ -\ln \frac{1 + \frac{\gamma - 1}{2} y}{y} \right\} \right]_{y_1}^{y_2} \\
 &= \frac{1}{\gamma} \left[\frac{-1}{M^2} + \left(1 + \frac{\gamma - 1}{2}\right) \ln \frac{1 + \frac{\gamma - 1}{2} M^2}{M^2} \right]_{M_1}^{M_2}
 \end{aligned}$$

Integration Result

- Combine results into expression for M change caused by friction

$$\frac{\bar{f} L}{D} = \left[\frac{-1}{\gamma M^2} + \left(\frac{\gamma+1}{2\gamma} \right) \ln \frac{1 + \frac{\gamma-1}{2} M^2}{M^2} \right]_{M_1}^{M_2}$$



(X.11)

- For example, given fL/D and M_1
 - could “solve” X.11 for M_2
- Can’t invert X.11 analytically - can’t write $M_2 = f(M_1, fL/D)$
 - either use *iterative* (e.g., numerical or guessing) method
 - or find fL_{max}/D as a function of M and tabularize solution

Use of Tables and Reference State

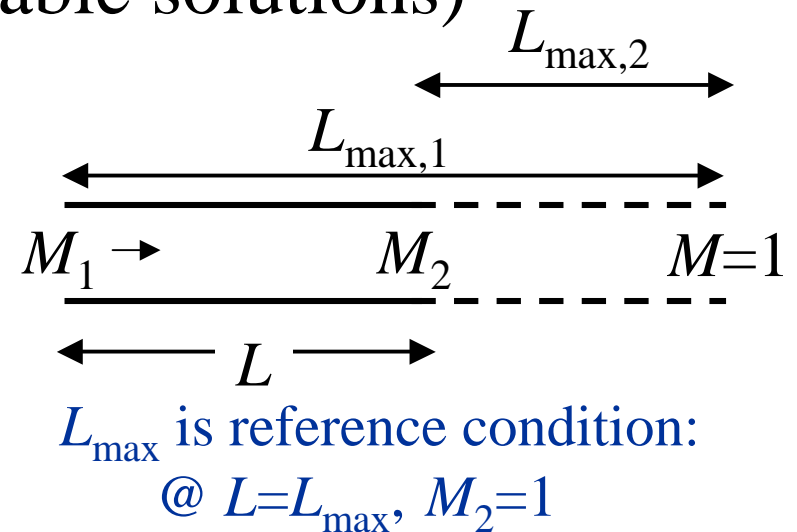
- To get change in M , use $M=1$ as reference condition (like Prandtl-Meyer and A/A^* table solutions)

$$\frac{fL}{D} = \left[\frac{-1}{\gamma M^2} + \left(\frac{\gamma+1}{2\gamma} \right) \ln \frac{1 + \frac{\gamma-1}{2} M^2}{M^2} \right]_{M_1}^{M_2}$$

$$\frac{fL}{D} = \left[\right]_{M_1}^1 - \left[\right]_{M_2}^1$$

(X.12)

$$\frac{fL}{D} = \frac{fL_{\max}}{D} \Big|_{M_1} - \frac{fL_{\max}}{D} \Big|_{M_2}$$



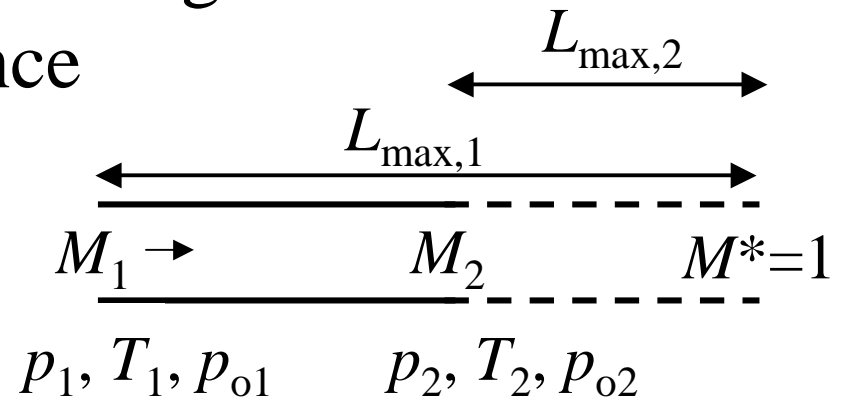
so if you know fL/D and M_1 ,

- 1) look up fL_{\max}/D at M_1
- 2) calculate fL_{\max}/D at M_2
- 3) look up corresponding M_2

- Find values in Appendix E in John

TD Property Changes

- To get changes in T, p, p_o, \dots can again use $M=1$ condition as reference condition (denoted as $*$)



- Integrate (X.7-10), e.g.,

$$\int_{p_1}^{p_2} \frac{dp}{p} = \int_{M_1}^{M_2} -\frac{1}{2} \frac{1 + (\gamma - 1)M^2}{1 + \frac{\gamma - 1}{2}M^2} \frac{dM^2}{M^2}$$

$$\frac{p_2}{p_1} = \left[\frac{M_1^2 \left(1 + \frac{\gamma - 1}{2} M_1^2 \right)}{M_2^2 \left(1 + \frac{\gamma - 1}{2} M_2^2 \right)} \right]^{1/2} \Rightarrow \frac{p}{p^*} = \frac{1}{M} \sqrt{\frac{\frac{\gamma + 1}{2}}{1 + \frac{\gamma - 1}{2} M^2}}$$

Fanno Flow Property Changes

- Summarize results in terms of **reference conditions**

$$\frac{T}{T^*} = \frac{(\gamma + 1)/2}{1 + \frac{\gamma - 1}{2} M^2} \quad (\text{X.13})$$

$$\frac{p_o}{p_o^*} = \frac{1}{M} \left(\frac{T}{T^*} \right)^{\frac{\gamma + 1}{2(1 - \gamma)}} \quad (\text{X.15})$$

$$\frac{p}{p^*} = \frac{1}{M} \sqrt{\frac{T}{T^*}} \quad (\text{X.14})$$

$$\frac{v}{v^*} = \frac{\rho^*}{\rho} = M \sqrt{\frac{T}{T^*}} \quad (\text{X.15})$$

- OR** in terms of **initial and final properties**

$$\frac{T_2}{T_1} = \frac{\left(1 + \frac{\gamma - 1}{2} M_1^2 \right)}{\left(1 + \frac{\gamma - 1}{2} M_2^2 \right)} \quad (\text{X.17}) \quad (T_o = \text{const})$$

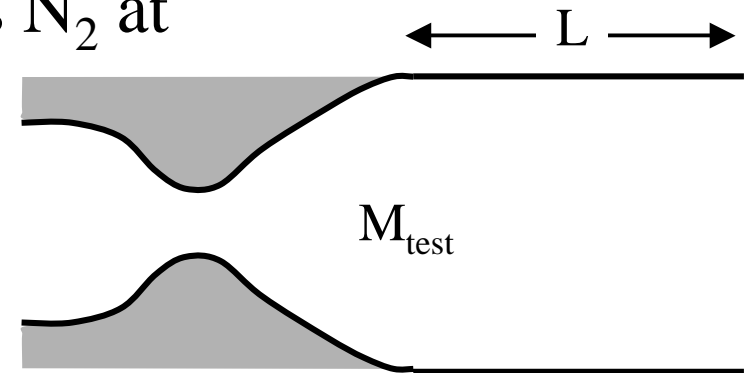
$$\frac{p_2}{p_1} = \frac{M_1}{M_2} \sqrt{\frac{T_2}{T_1}} \quad (\text{X.18})$$

$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2} \left(\frac{T_2}{T_1} \right)^{\frac{\gamma + 1}{2(1 - \gamma)}} \quad (\text{X.19})$$

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1} \sqrt{\frac{T_2}{T_1}} \quad (\text{X.20})$$

Example

- Given:** Exit of supersonic nozzle connected to straight walled test section. Test section flows N_2 at $M_{\text{test}}=3.0$, $T_o=290$. K, $p_o=500$. kPa, $L=1$ m, $D=10$ cm, $f=0.005$

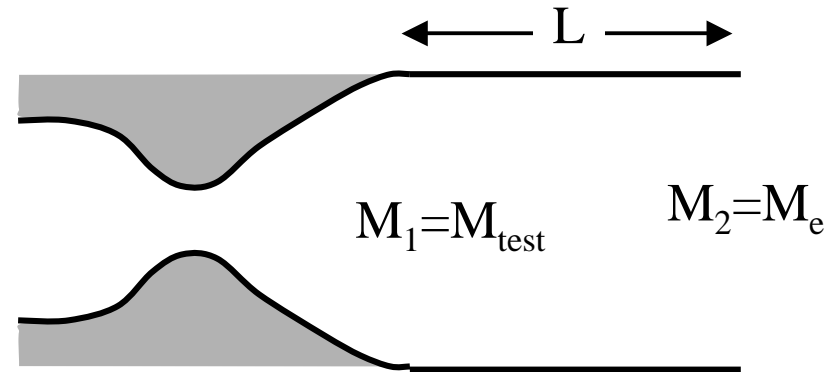


- Find:**
 - M , T , p at end of test section
 - $P_{o,\text{exit}}/P_{o,\text{inlet}}$
 - L_{max} for test section
- Assume:** N_2 is tpg/cpg, $\gamma=1.4$, steady, adiabatic, no work

Solution

- Analysis:

$$- \mathbf{M}_e \left(\frac{fL}{D} = \frac{fL_{\max}}{D} \right)_{3.0} - \left(\frac{fL_{\max}}{D} \right)_{M_e}$$



$$\left(\frac{fL_{\max}}{D} \right)_{M_e} = 0.5222 - \frac{0.005(100)}{10} = 0.4722$$

(Appendix E) \rightarrow $M_e = 2.70$

another solution is $M=0.605$, but since started $M>1$, can't be subsonic

$$- \mathbf{T} \quad (T_o \text{ const}) \quad T_2 = T_1 \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2} = \frac{T_o}{1 + \frac{\gamma-1}{2} M_2^2} = 118 \text{ K}$$

Solution (con't)

– **p**
$$p_2 = p_1 \frac{M_1}{M_2} \sqrt{\frac{T_2}{T_1}} \quad (\text{X.17})$$

$$p_1 = p_{o1} \left(1 + \frac{\gamma-1}{2} M_1^2 \right)^{-\gamma/\gamma-1}$$

$$= \frac{500 \text{ kPa}}{2.8^{3.5}} = 13.6 \text{ kPa}$$

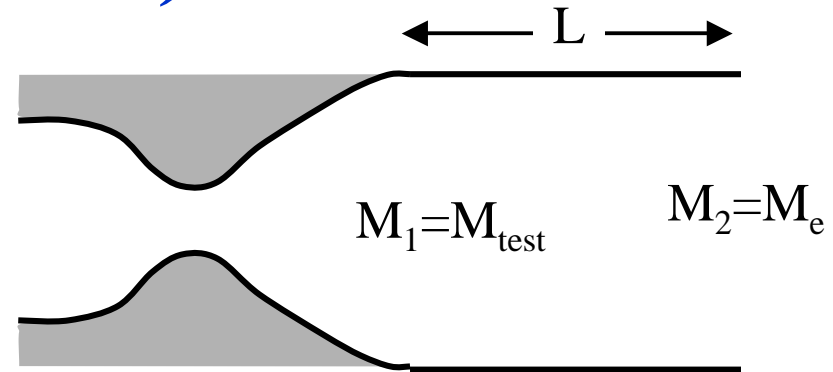
$$\frac{T_2}{T_1} = \frac{1 + ((\gamma-1)/2)M_1^2}{1 + ((\gamma-1)/2)M_2^2} = 1.14$$

$$p_2 = 13.6 \text{ kPa} \frac{3.0}{2.7} \sqrt{1.14} = 16.1 \text{ kPa}$$

– **$p_{o,e}/p_{o,test}$**

$$(\text{X.19}) \quad \frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2} \left(\frac{T_2}{T_1} \right)^{\frac{\gamma+1}{2(1-\gamma)}} = \frac{3.0}{2.7} (1.14)^{-3} = 0.75$$

25% loss in stagnation pressure due to friction



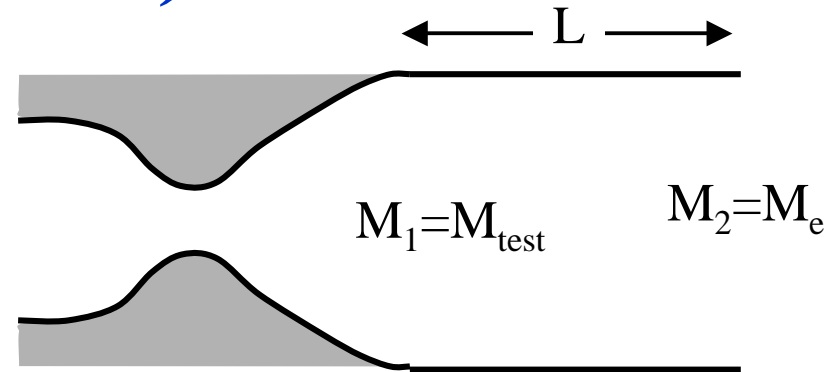
Solution (con't)

– L_{\max}

$$L_{\max} = \left(\frac{f L_{\max}}{D} \right)_{M_{\text{test}}} \frac{D}{f}$$

$$= 0.5222 \frac{0.1\text{m}}{0.005}$$

$$= 10.4\text{m} \quad \text{10 m long section would have } M=1 \text{ at exit}$$



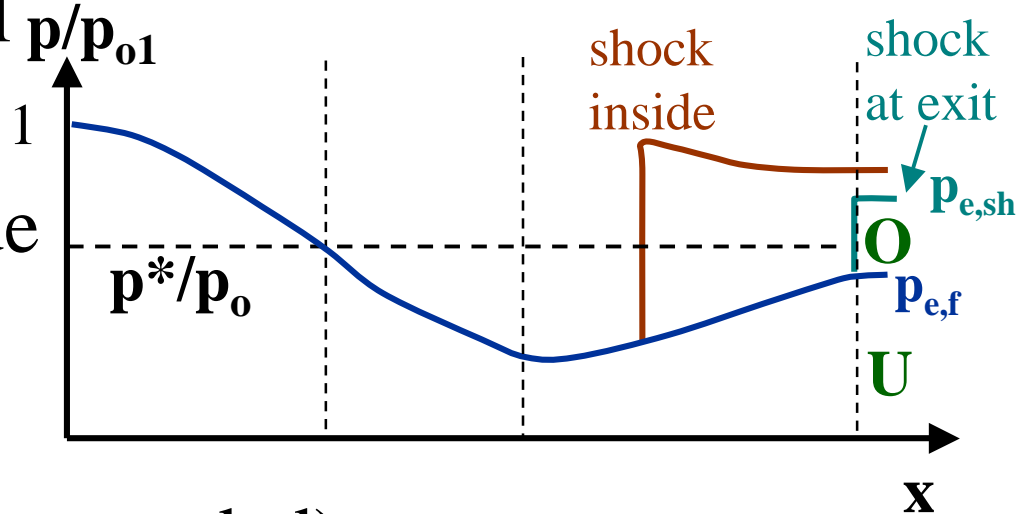
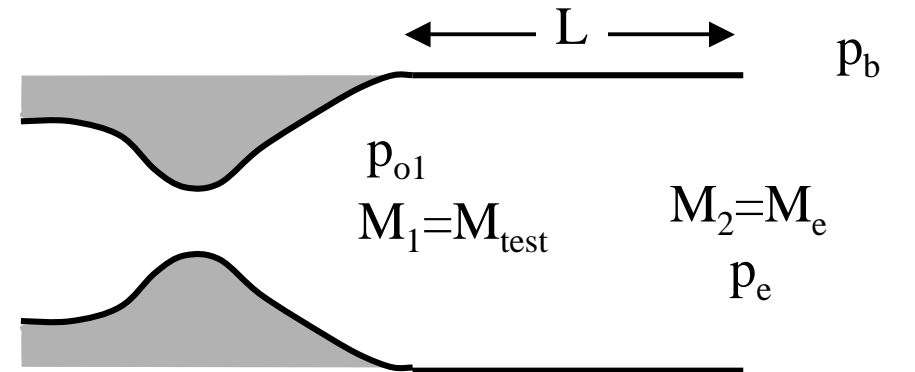
$L < L_{max}$, Back Pressure

- Last problem (supersonic duct), what would happen if calculated exit pressure ($p_{e,f}$) did not match actual back pressure (p_b)

- $p_b < p_{e,f}$: expansion outside duct (underexpanded)

- $p_{e,f} < p_b < p_{e,sh}$: oblique shocks outside duct (overexpanded)

- $p_{e,sh} < p_b$: shocks inside duct (until shock reaches ~throat)



$L > L_{max}$, Back Pressure

- Can't have flow transition to subsonic with pure Fanno flow
 \Rightarrow shock in duct

- Shock location fixed by back pressure
 - low enough p_b , $M_e=1$
 - raise p_b , shock moves upstream until it reaches *sonic* location in nozzle

