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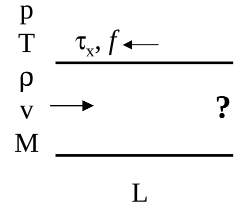
Fanno Flow - Thermodynamics



- Steady, 1-d, constant area, adiabatic flow with no external work but *with friction*
- Conserved quantities
  - since adiabatic, no work: **h_o=constant**
  - since A=const: mass flux=**ρv=constant≡G**
  - combining: $h_o = h + G^2/2\rho^2$=constant

As you change h, you change ρ (and v) since G and h_o const.

- On h-s diagram, can draw **Fanno Line**
  - line connecting points with same h_o and ρv

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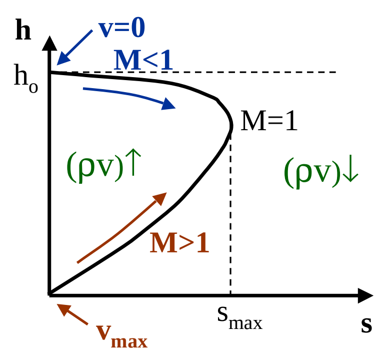
Fanno Line

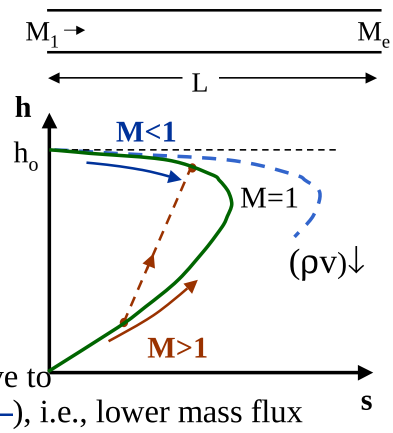
- Velocity change (due to friction) associated with entropy change
- Friction can only increase entropy
  - can only approach M=1
  - friction alone can not allow flow to transition between sub/supersonic



- Two solutions given (ρv,h_o,s): subsonic & supersonic
  - change mass flux: new Fanno line

Fanno Line - Choking

- Total friction experienced by flow increases with length of "flow", e.g., duct length, L
- For long enough duct, M_e=1 ($L=L_{max}$)
- What happens if $L>L_{max}$
  - flow already "choked"
  - **subsonic flow**: must move to different Fanno line (– – –), i.e., lower mass flux
  - **supersonic flow**: get a shock (– – –)



Fanno Line – Mach Equations

- Simplify (X.4-5) for δq=dA=0

$$\frac{dM^2}{M^2}=\frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\frac{f dx}{D} \quad \textbf{(X.6)}$$

$$\frac{dT}{T}=\frac{dh}{h}=\frac{-\gamma(\gamma-1)M^4}{2(1-M^2)}\frac{f dx}{D} \quad \textbf{(X.8)}$$

$$\frac{dp}{p}=\frac{-\gamma M^2\left[1+(\gamma-1)M^2\right]}{2(1-M^2)}\frac{f dx}{D} \quad \textbf{(X.7)}$$

$$\frac{d\rho}{\rho}=-\frac{dv}{v}=\frac{-\gamma M^2}{2(1-M^2)}\frac{f dx}{D} \quad \textbf{(X.9)}$$

$$\frac{ds}{R}=-\frac{dp_o}{p_o}=\frac{\gamma M^2}{2}\frac{f dx}{D} \quad \textbf{(X.10)}$$

- can write each as only f(M)
- p_o loss due to entropy rise

Property Variations

- Look at signs of previous equations to see how properties changed by friction as we move along flow
  - $(1-M^2)$ term makes M<1 different than M>1

	M<1	M>1	
s	↑	↑	• Friction increases s, ⇒p_o drop
p_o	↓	↓	
M	↑	↓	• Friction drives M→1
h,T	↓	↑	• h_o,T_o const: h,T opposite to M
p	↓	↑	• p, ρ same as T (like isen. flow)
ρ	↓	↑	
v	↑	↓	•ρv=const: v opposite of ρ

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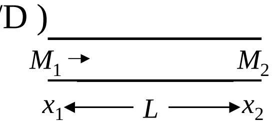
Integration

- Need to integrate (X.6-10) to find how properties change along length of flow (*f*dx/D)
  - for example, need to integrate



$$\frac{dM^2}{M^2}=\frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\frac{fdx}{D}$$

- Separate terms

$$\int_{M_1}^{M_2}\frac{\left(1-M^2\right)dM^2}{\gamma M^4\left(1+\frac{\gamma-1}{2}M^2\right)}=\int_{x_1}^{x_2}\frac{f\left(\text{Re, surface}\right)dx}{D}$$

f function of Reynolds number (e.g.,velocity) and surface roughness

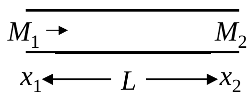
$$\int_{x_1}^{x_2}\frac{f\left(\text{Re, surface}\right)dx}{D}\cong\frac{\bar{f}\left(x_2-x_1\right)}{D}=\frac{\bar{f}\,L}{D}$$

for simplicity, can approximate *f* by average value

Mach Number Integral

- To perform M integral, redefine variables

$$\int_{M_1}^{M_2} \frac{\left(1-M^2\right)dM^2}{\gamma M^4\left(1+\frac{\gamma-1}{2}M^2\right)} = \int_{y_1}^{y_2}\frac{(1-y)dy}{\gamma y^2\left(1+\frac{\gamma-1}{2}y\right)}$$



$$=\frac{1}{\gamma}\left[\left\{\int_{y_1}^{y_2}\frac{dy}{y^2\left(1+\frac{\gamma-1}{2}y\right)}\right\}-\left\{\int_{y_1}^{y_2}\frac{dy}{y\left(1+\frac{\gamma-1}{2}y\right)}\right\}\right]$$

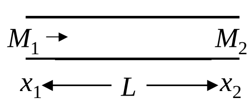
$$=\frac{1}{\gamma}\left[\left\{\frac{-1}{y}+\frac{\gamma-1}{2}\ln\frac{1+\frac{\gamma-1}{2}y}{y}\right\}-\left\{-\ln\frac{1+\frac{\gamma-1}{2}y}{y}\right\}\right]_{y_1}^{y_2}$$

$$=\frac{1}{\gamma}\left[\frac{-1}{M^2}+\left(1+\frac{\gamma-1}{2}\right)\ln\frac{1+\frac{\gamma-1}{2}M^2}{M^2}\right]_{M_1}^{M_2}$$

Integration Result

- Combine results into expression for *M* change caused by friction

$$\frac{\bar{f}\,L}{D}=\left[\frac{-1}{\gamma M^2}+\left(\frac{1+\gamma}{2\gamma}\right)\ln\frac{1+\frac{\gamma-1}{2}M^2}{M^2}\right]_{M_1}^{M_2}$$



(X.11)

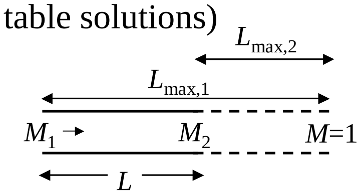
- For example, given $f\,L/D$ and M_1
  - could "solve" X.11 for M_2
- Can't invert X.11 analytically - can't write M_2= f(M_1, $f\,L/D$)
  - either use *iterative* (e.g., numerical or guessing) method
  - or find *fL_{max}/D as a function of M* and tabularize solution

Use of Tables and Reference State

- To get change in M, use M=1 as reference condition (like Prandtl-Meyer and A/A^* table solutions)

$$\frac{f\,L}{D}=\left[\frac{-1}{\gamma M^2}+\left(\frac{1+\gamma}{2\gamma}\right)\ln\frac{1+\frac{\gamma-1}{2}M^2}{M^2}\right]_{M_1}^{M_2}$$

$$\frac{fL}{D}=\Big[\quad\Big]_{M_1}^{1}-\Big[\quad\Big]_{M_2}^{1}$$



L_{max} is reference condition: @ $L=L_{max}$, M_2=1

(X.12)

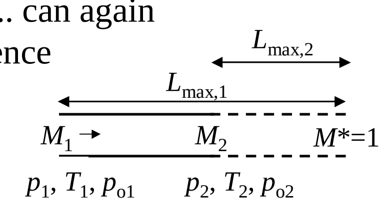
$$\frac{fL}{D}=\left.\frac{fL_{max}}{D}\right|_{M_1}-\left.\frac{fL_{max}}{D}\right|_{M_2}$$

so if you know f L/D and M_1,
1) look up $f\,L_{max}/D$ at M_1
2) calculate $f\,L_{max}/D$ at M_2
3) look up corresponding M_2

- Find values in Appendix E in John

TD Property Changes

- To get changes in T, p, p_o, ... can again use M=1 condition as reference condition (denoted as *)



- Integrate (X.7-10), e.g.,

$$\int_{p_1}^{p_2}\frac{dp}{p}=\int_{M_1}^{M_2}-\frac{1}{2}\frac{1+(\gamma-1)M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2}$$

$$\frac{p_2}{p_1}=\left[\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}{M_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)}\right]^{1/2}\Rightarrow\frac{p}{p^*}=\frac{1}{M}\sqrt{\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}}$$

Fanno Flow Property Changes

- Summarize results in terms of reference conditions

$$\frac{T}{T^*} = \frac{(\gamma+1)/2}{1+\frac{\gamma-1}{2}M^2} \quad \textbf{(X.13)}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M}\left(\frac{T}{T^*}\right)^{\frac{\gamma+1}{2(1-\gamma)}} \quad \textbf{(X.15)}$$

$$\frac{p}{p^*} = \frac{1}{M}\sqrt{\frac{T}{T^*}} \quad \textbf{(X.14)}$$

$$\frac{v}{v^*} = \frac{\rho^*}{\rho} = M\sqrt{\frac{T}{T^*}} \quad \textbf{(X.15)}$$

- **OR** in terms of initial and final properties

$$\frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)} \quad \textbf{(X.17)}\ (T_o=\text{const})$$

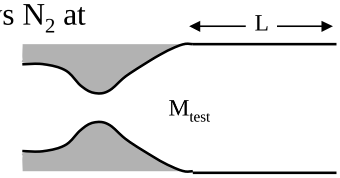
$$\frac{p_2}{p_1} = \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}} \quad \textbf{(X.18)}$$

$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma+1}{2(1-\gamma)}} \quad \textbf{(X.19)}$$

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \quad \textbf{(X.20)}$$

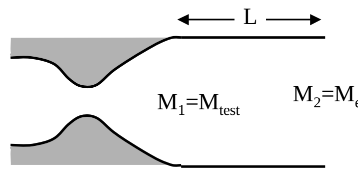
Example

- **Given:** Exit of supersonic nozzle connected to straight walled test section. Test section flows N_2 at M_{test}=3.0, T_o=290. K, p_o=500. kPa, L=1m, D=10 cm, *f*=0.005



- **Find:**
  - M, T, p at end of test section
  - $p_{o,exit}/p_{o,inlet}$
  - L_{max} for test section
- **Assume:** N_2 is tpg/cpg, γ=1.4, steady, adiabatic, no work

Solution



- **Analysis:**
  - **M_e** (X.11) $\frac{fL}{D} = \left.\frac{fL_{max}}{D}\right)_{3.0} - \left.\frac{fL_{max}}{D}\right)_{M_e}$

$$\left.\frac{fL_{max}}{D}\right)_{M_e} = 0.5222 - \frac{0.005(100)}{10} = 0.4722$$

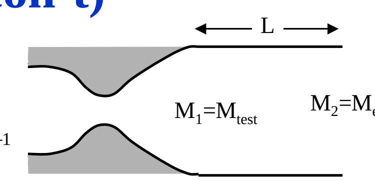
(Appendix E) → $M_e = 2.70$

another solution is M=0.605, but since started M>1, can't be subsonic

  - **T**

(T$_o$ const) $$T_2 = T_1 \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} = \frac{T_o}{1+\frac{\gamma-1}{2}M_2^2} = 118\,K$$

Solution (con't)



- **p**

$$p_2 = p_1 \frac{M_1}{M_2}\sqrt{\frac{T_2}{T_1}} \quad \text{(X.17)}$$

$$p_1 = p_{o1}\left(1+\frac{\gamma-1}{2}M_1^2\right)^{-\gamma/\gamma-1}$$

$$= \frac{500\,kPa}{2.8^{3.5}} = 13.6kPa \qquad \frac{T_2}{T_1} = \frac{1+((\gamma-1)/2)M_1^2}{1+((\gamma-1)/2)M_2^2} = 1.14$$

$$p_2 = 13.6kPa\,\frac{3.0}{2.7}\sqrt{1.14} = 16.1kPa$$

- **$p_{o,e}/p_{o,test}$**

(X.19) $$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2}\left(\frac{T_2}{T_1}\right)^{\frac{\gamma+1}{2(1-\gamma)}} = \frac{3.0}{2.7}(1.14)^{-3} = 0.75$$

25% loss in stagnation pressure due to friction

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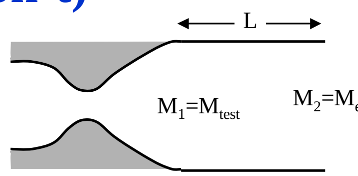
Solution (con't)

– L_{max}



$$L_{max} = \left.\frac{fL_{max}}{D}\right)_{M_{test}} \frac{D}{f}$$

$$= 0.5222 \frac{0.1m}{0.005}$$

$$= 10.4m$$

10 m long section would have M=1 at exit

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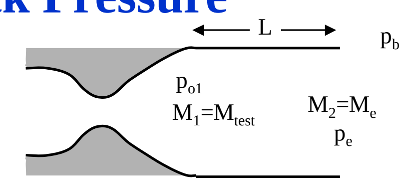
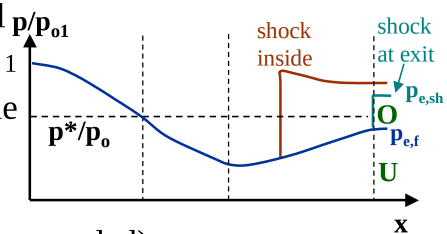
$L<L_{max}$, Back Pressure



- Last problem (supersonic duct), what would happen if calculated exit pressure ($p_{e,f}$) did not match actual back pressure (p_b)



- $p_b<p_{e,f}$: expansion outside duct (underexpanded)
- $p_{e,f} < p_b < p_{e,sh}$: oblique shocks outside duct (overexpanded)
- $p_{e,sh}<p_b$: shocks inside duct (until shock reaches ~throat)

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L>L$_{max}$, Back Pressure

- Can't have flow transition to subsonic with pure Fanno flow
  ⇒shock in duct
- Shock location fixed by back pressure
  - low enough p_b, M_e=1
  - raise p_b, shock moves upstream until it reaches *sonic* location in nozzle