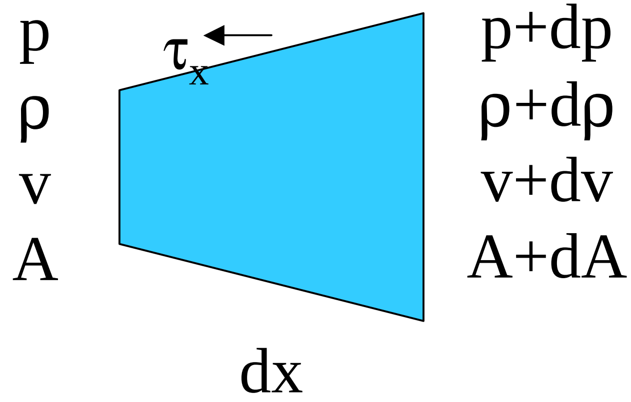
Isentropic Flow with Area Change

- Examine mass and momentum equations for **reversible** and **adiabatic** conditions



Mass (VI.9)

$$\frac{d\rho}{\rho}+\frac{1}{2}\frac{dv^2}{v^2}+\frac{dA}{A}=0$$

Momentum (VI.10)

$$\tau_x\frac{L_p}{A}dx+\frac{dp}{p}+\frac{1}{2}\frac{v^2}{p/\rho}\frac{dv^2}{v^2}=0\Rightarrow\boxed{\frac{1}{2}\frac{dv^2}{v^2}=-\frac{dp}{\rho v^2}}\quad *$$

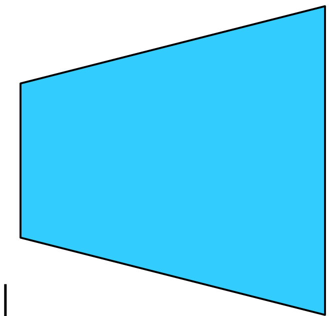
0 (no viscous stress/friction, reversible)

- Combine

$$\frac{d\rho}{\rho}-\frac{dp}{\rho v^2}+\frac{dA}{A}=0\Rightarrow\frac{dA}{A}=\frac{dp}{\rho v^2}\left(1-\frac{v^2}{dp/d\rho}\right)$$

Mach Number Relation

$$\frac{dA}{A} = \frac{dp}{\rho v^2}\left(1 - \frac{v^2}{dp/d\rho}\right)$$



- But isentropic (rev./adiab.), so $\frac{dp}{d\rho} = \left.\frac{\partial p}{\partial \rho}\right|_s = a^2$

$$\frac{dA}{A} = \frac{dp}{\rho v^2}\left(1 - M^2\right) = -\frac{dv}{v}\left(1 - M^2\right)$$ **(VI.16)**

From * $$\frac{dp}{\rho v^2} = -\frac{1}{2}\frac{dv^2}{v^2} = -\frac{dv}{v}$$

- **Derived using only mass/momentum conservation, and speed of sound**
- **Valid for all simple comp. substances**

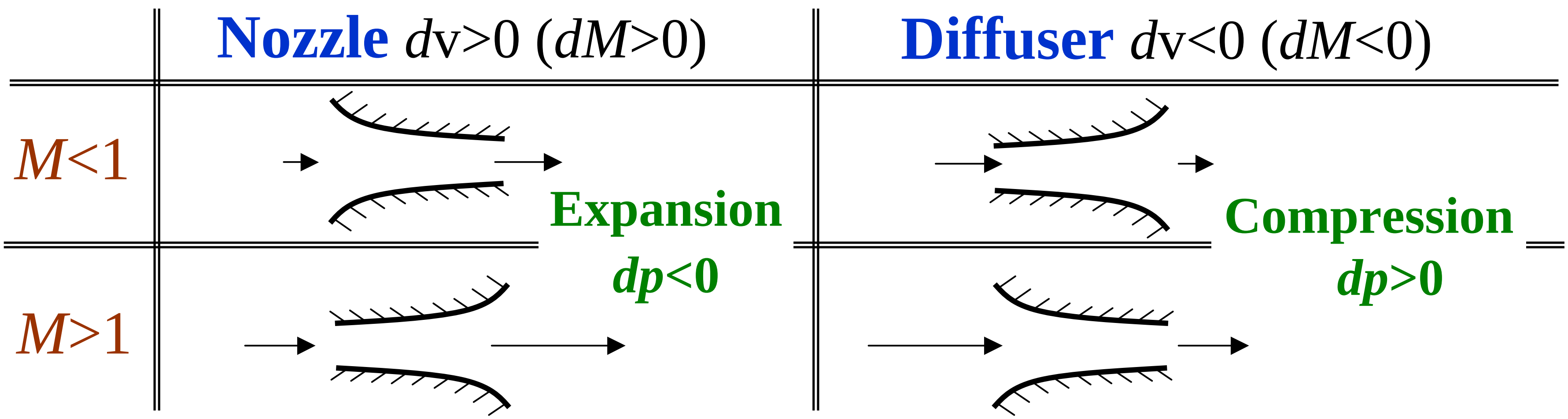
Isen. Flow - Mach Number Dependence

$$\frac{dA}{A}=\frac{dp}{\rho \mathrm{v}^2}\left(1-M^2\right)=-\frac{d\mathrm{v}}{\mathrm{v}}\left(1-M^2\right)$$

- **How does area change effect flow properties?**

- For M<1: dA, dp same sign — $A\uparrow \Rightarrow p\uparrow$
  dA, $d\mathrm{v}$ opposite sign — $A\uparrow \Rightarrow \mathrm{v}\downarrow$
- For M>1: dA, dp opposite sign — $A\uparrow \Rightarrow p\downarrow$
  dA, $d\mathrm{v}$ same sign — $A\uparrow \Rightarrow \mathrm{v}\uparrow$

- dp, $d\mathrm{v}$ always opposite signs ($\mathrm{v}\uparrow \Rightarrow T\downarrow \Rightarrow p\downarrow$)



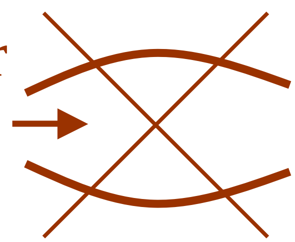
Sonic Throat Requirement

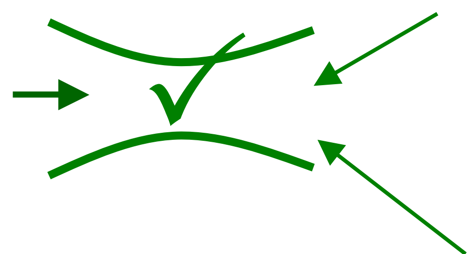
$$\frac{dA}{A}=\frac{dp}{\rho v^2}\left(1-M^2\right)=-\frac{dv}{v}\left(1-M^2\right)$$

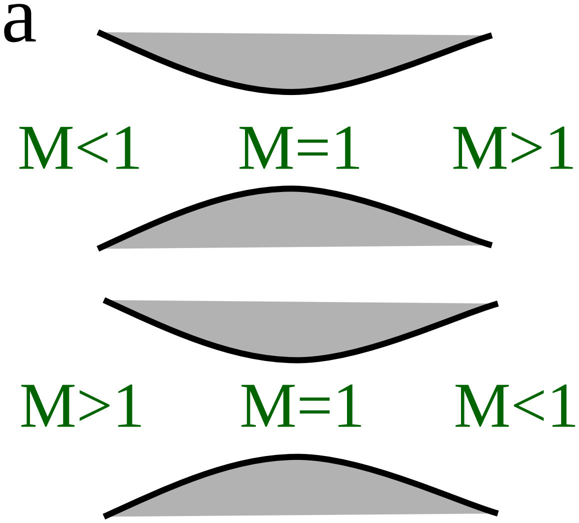
- **How to transition from subsonic to supersonic (or vica versa)?**

- Need to go through M=1
- For M=1: dA or dv = 0
  but dv≠0 (flow accel. or decel.)
- dA=0: maximum or minimum in area

M<1⇒M gets smaller
M>1⇒ M gets bigger
then never get M=1



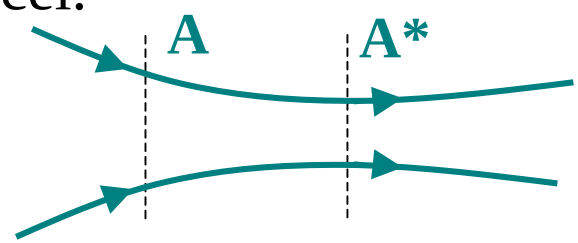




- So **need a throat** to transition,
  and **M=1 at throat** (sonic condition)

Area Ratio

- For isentropic flow, look at effect of area change on M by comparing A at any point to area at sonic point (A*)
  - * refers to properties of a flow if isen. accel./decel. to M=1 sonic conditions (e.g., ρ*, T*,…)
  - alternative to stagnation as ref. state
- Use mass conservation to find relation



$$\rho v A = \rho^* v^* A^*$$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{v^*}{v} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{\cancelto{a^*}{v^*}}{a}\cancelto{\frac{1}{M}}{\frac{a}{v}}$$

$$\frac{\rho_o}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{1/\gamma-1}$$ from VI.8

$$\frac{\rho_o}{\rho^*} = \left(1+\frac{\gamma-1}{2}\right)^{1/\gamma-1}$$

$$\frac{a^*}{a} = \sqrt{\frac{T^*}{T}} = \sqrt{\frac{T^*}{T_o}\frac{T_o}{T}}$$ with $\frac{T_o}{T} = 1+\frac{\gamma-1}{2}M^2$, $\frac{T_o}{T^*} = 1+\frac{\gamma-1}{2}$ from VI.6

$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left\{\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right\}^{\gamma+1/2(\gamma-1)}}$$

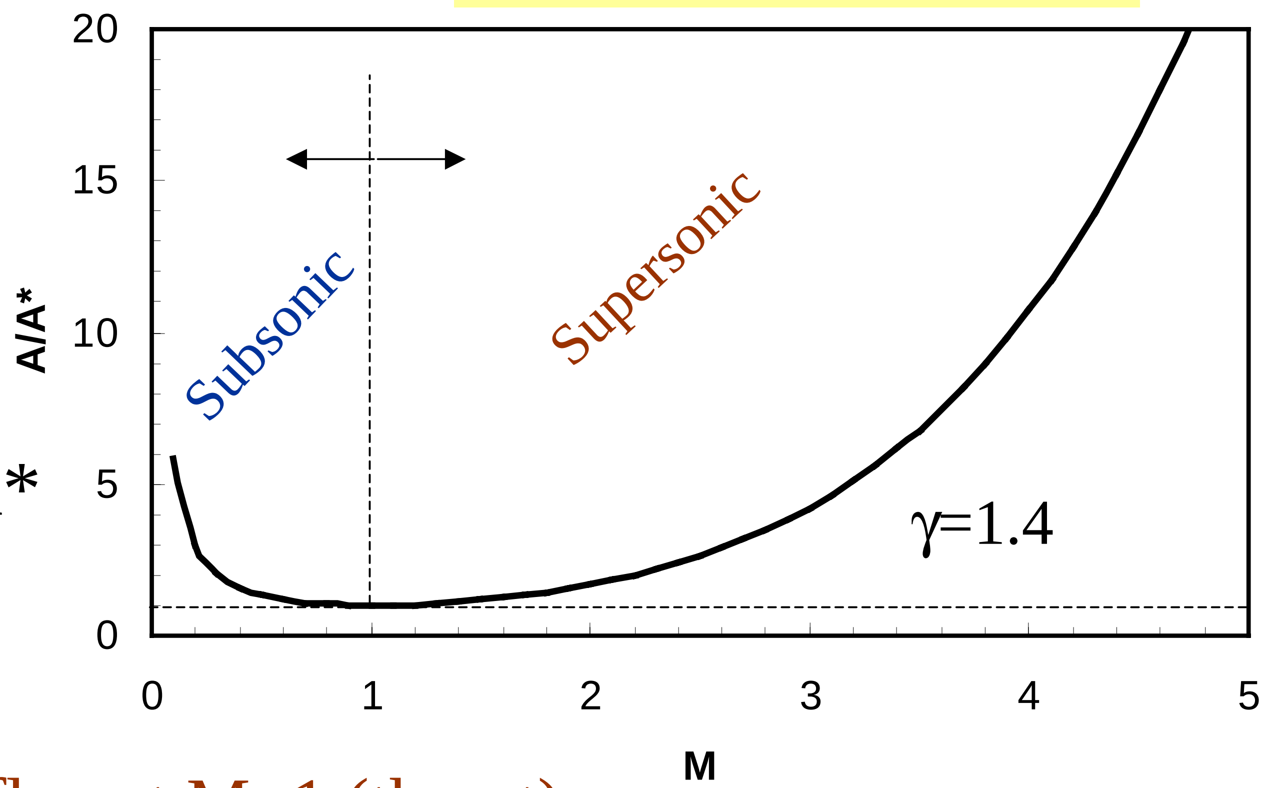
(VI.17)

for TPG, CPG

Area Ratio Results

$$\frac{A}{A^*} = \frac{1}{M}\left\{\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right\}^{\frac{\gamma+1}{2(\gamma-1)}}$$

- Two (isentropic) solutions for a given A/A*
  - one subsonic
  - one supersonic
- A≥A* always
- Accel. to high M requires large A/A*
- $\dfrac{\dot{m}/A}{(\dot{m}/A)^*} = \dfrac{\rho v}{\rho^* v^*} = \dfrac{A^*}{A}$
  - maximum mass flux at M=1 (throat)



Mass Flux and Stagnation Properties

- Examine mass flux in terms of stagnation conditions

from VI.2, **TPG**

$$\frac{\dot{m}}{A} = \rho v = \frac{p}{RT} Ma = \frac{p}{RT} M\sqrt{\gamma RT}$$

from VI.6,7 **CPG**

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma-1}$$

$$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}} \frac{\sqrt{\gamma} M}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma+1/2(\gamma-1)}} \quad \textbf{(VI.18)}$$

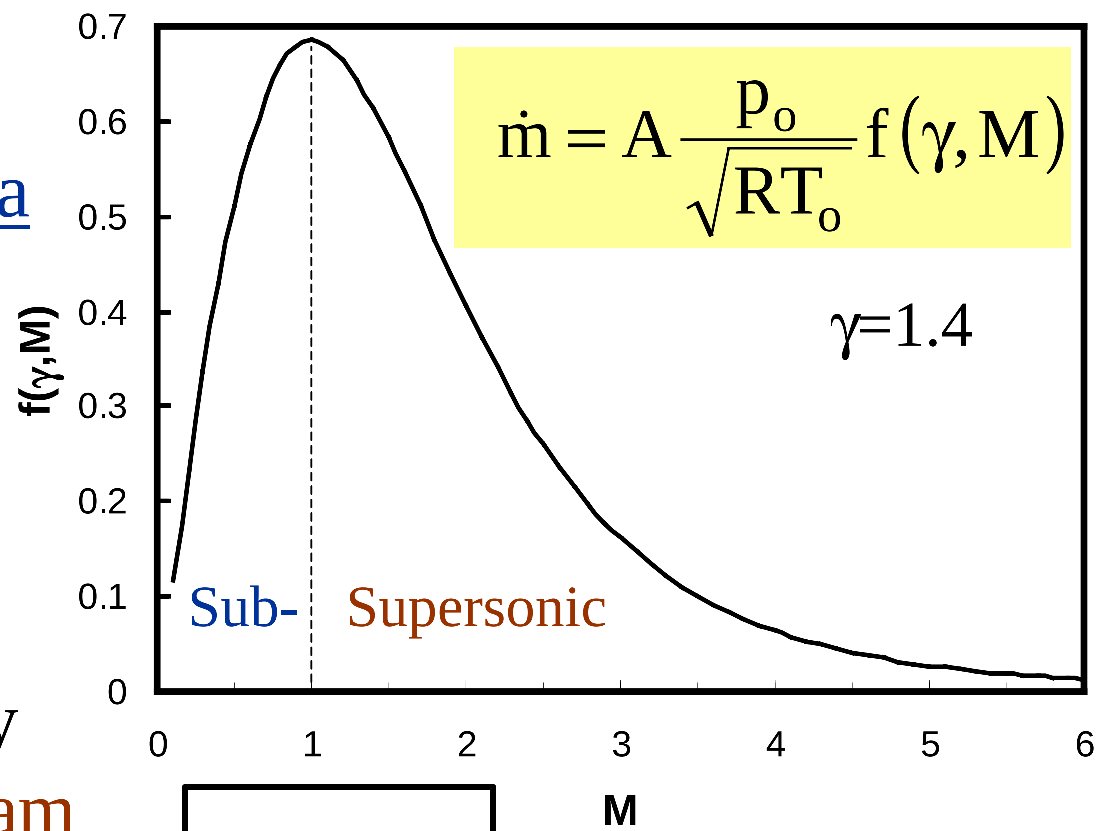
or $$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}} f(\gamma, M)$$

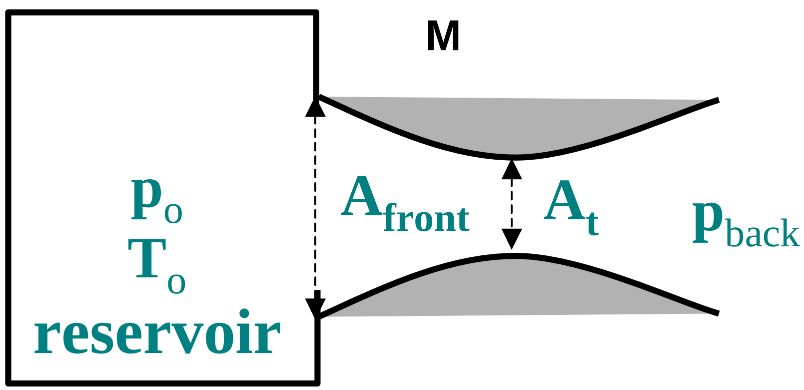
- For given isen. flow, all stagnation (and sonic) properties constant, including mass flow rate

Choked Flow

- For fixed stagnation properties and flow area
  - $\max\ \dot{m}$ at $M = 1$
- For nozzle with fixed stagnation properties
  - if throat is sonic, can't get more $\dot{m}$ by changing downstream conditions (e.g., back pressure)

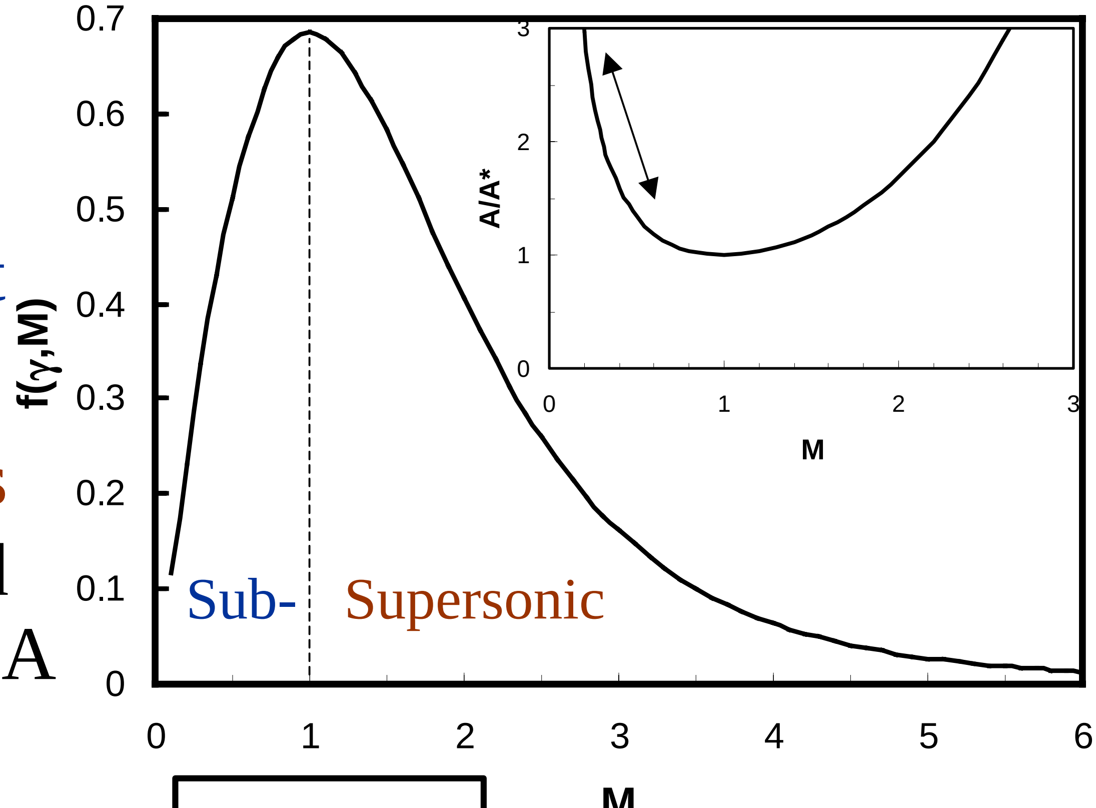
    $\Rightarrow$ **Choked Flow**

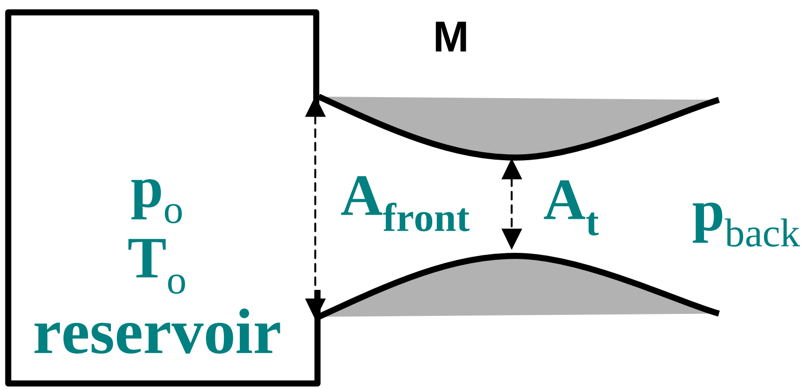




Choked Flow (con't)

- For nozzle with **fixed stagnation properties** and initially sonic throat
  - if reduce throat area A_t, flow at throat stays sonic ($A_{front}/A_t \uparrow$) and $\dot{m}\downarrow$ (since $A\downarrow$, & $\dot{m}/A$ same)
  - if increase A_t, $\dot{m}\uparrow$ and eventually throat not sonic ($A_{front}/A^* \downarrow$) (not choked and $\dot{m} \neq \dot{m}_{max}$)





Choked Mass Flowrate

- Maximum flow rate when choked (M=1 at throat)
- Choked mass flowrate from f(γ,1)

$$\dot{m}_{max} = A^* \frac{p_o}{\sqrt{RT_o}} \sqrt{\gamma} \left(1 + \frac{\gamma - 1}{2}\right)^{\gamma+1 / 2(1-\gamma)} \qquad \textbf{(VI.19)}$$

- To increase mass flowrate
  - increase A^* (throat size)
  - increase p_o, decrease T_o (increase stagnation density)
- f(γ,1) typically near 0.7

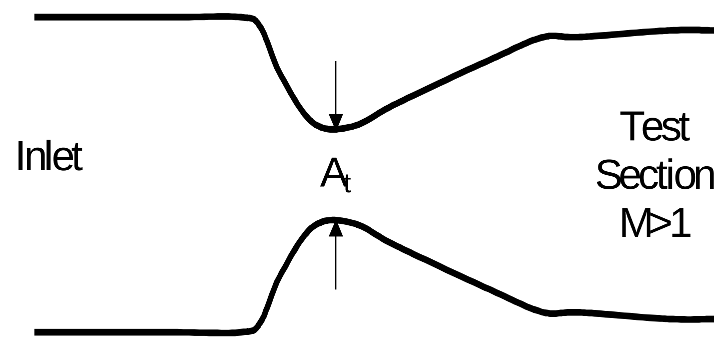
$$\dot{m}_{max} \approx 0.7 \frac{p_o}{\sqrt{RT_o}} A_{throat}$$

$$f(\gamma,1) = \begin{cases} 0.726 & \gamma = 5/3 \\ 0.685 & \gamma = 1.4 \\ 0.667 & \gamma = 1.3 \end{cases}$$

rule of thumb for **choked** gas flows

Example 1

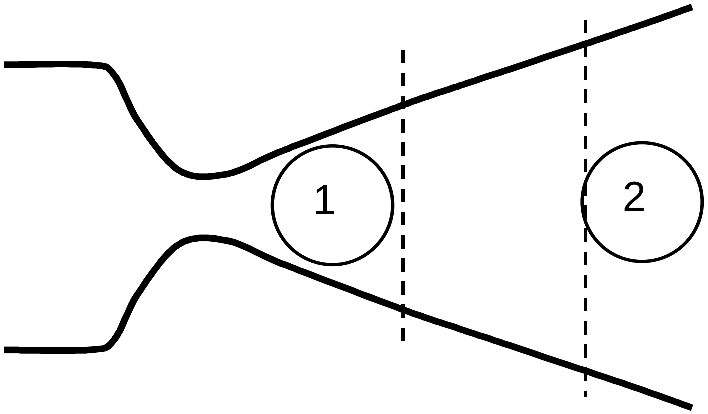
- For a supersonic wind tunnel with an incoming flow with M<<1 (from a high pressure reservoir or compressor), need a throat to produce M>1



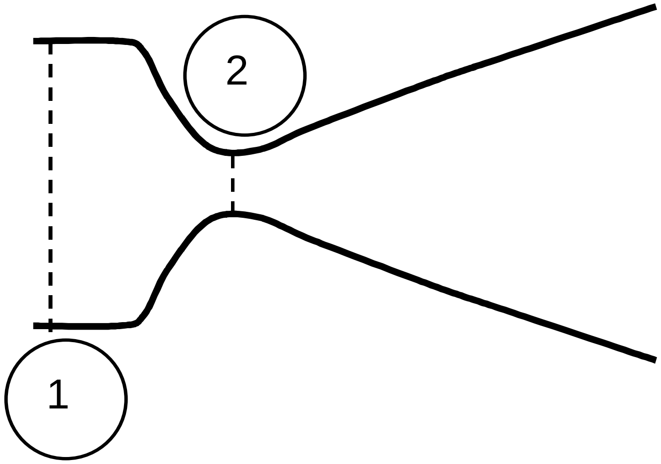
- What area throat required to produce a test section Mach number of M=3 in test section with 0.2 m^2 cross-section?
  - Assume isentropic flow, calor./thermally perfect gas, γ=1.4.

Example 2

- Converging-diverging supersonic nozzle with M_1=2.0 and M_2=3.0
- What is A_2/A_1?
  - Assume isentropic flow, calor./thermally perfect gas, γ=1.4.

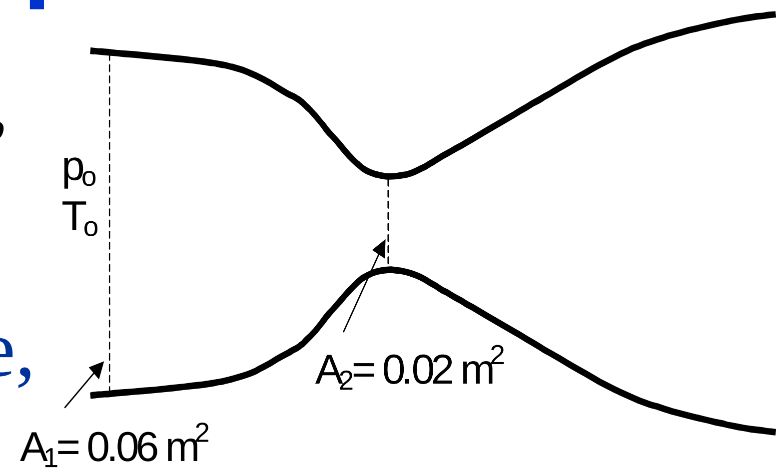


Example 3



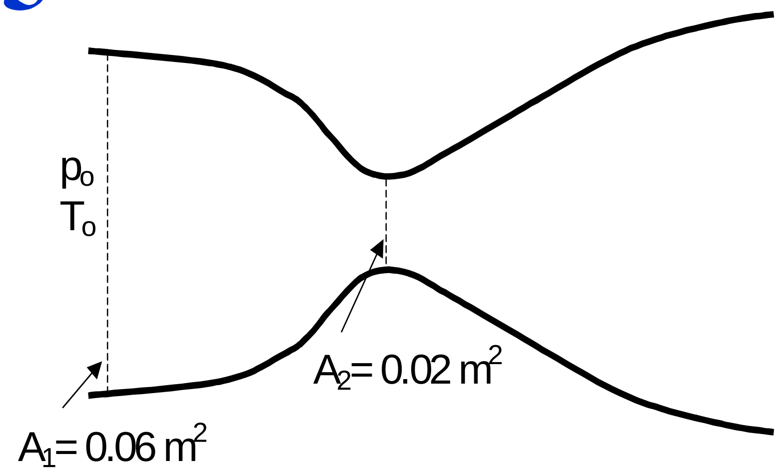
- For nozzle shown, inlet area A_1=0.50 m^2, M_1=0.20 and p_o, T_o fixed
- How far can A_2 be reduced from A_1 without changing mass flowrate in nozzle?
  - Assume isentropic flow, calor./thermally perfect gas, γ=1.4.

Example 4



- For nozzle shown (A_1=0.060m^2, A_2=0.020m^2) and p_o, T_o fixed
- For isentropic flow in the nozzle, what are the limits on the allowed inlet Mach numbers (M_1)?
  – Assume tpg/cpg, γ=1.4.

Example 5



- Same nozzle as Example 4, air and p_o=500kPa, T_o= 300K
- What is max. possible mass flowrate through the nozzle?
  - Assume tpg/cpg, γ=1.4.