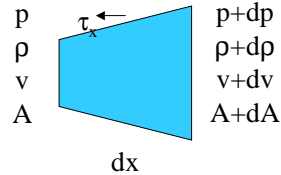


Isentropic Flow with Area Change

- Examine **mass** and **momentum** equations for **reversible** and **adiabatic** conditions



Mass (VI.9)
$$\frac{dp}{\rho} + \frac{1}{2} \frac{dv^2}{v^2} + \frac{dA}{A} = 0$$

Momentum (VI.10)
$$\int_x^x \frac{L_p}{A} dx + \frac{dp}{p} + \frac{1}{2} \frac{v^2}{p/\rho} \frac{dv^2}{v^2} = 0 \Rightarrow \frac{1}{2} \frac{dv^2}{v^2} = -\frac{dp}{\rho v^2} *$$

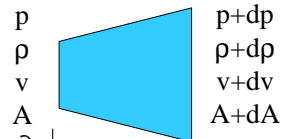
0 (no viscous stress/friction, reversible)

- Combine

$$\frac{dp}{\rho} - \frac{dp}{\rho v^2} + \frac{dA}{A} = 0 \Rightarrow \frac{dA}{A} = \frac{dp}{\rho v^2} \left(1 - \frac{v^2}{dp/d\rho} \right)$$

Mach Number Relation

$$\frac{dA}{A} = \frac{dp}{\rho v^2} \left(1 - \frac{v^2}{dp/d\rho} \right)$$



- But isentropic (rev./adiab.), so $\frac{dp}{d\rho} = \left. \frac{\partial p}{\partial \rho} \right|_s = a^2$

$$\frac{dA}{A} = \frac{dp}{\rho v^2} (1 - M^2) = -\frac{dv}{v} (1 - M^2) \quad \text{(VI.16)}$$

From *
$$\frac{dp}{\rho v^2} = -\frac{1}{2} \frac{dv^2}{v^2} = -\frac{dv}{v}$$

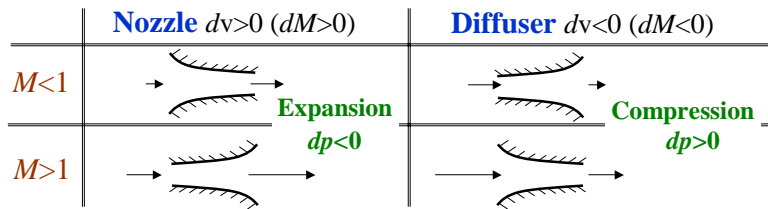
- Derived using only mass/momentum conservation, and speed of sound
- Valid for all simple comp. substances

Isen. Flow - Mach Number Dependence

$$\frac{dA}{A} = \frac{dp}{\rho v^2} (1 - M^2) = -\frac{dv}{v} (1 - M^2)$$

• How does area change effect flow properties?

- For $M < 1$: dA, dp same sign
 dA, dv opposite sign
 - For $M > 1$: dA, dp opposite sign
 dA, dv same sign
- $A \uparrow \Rightarrow p \uparrow$
 $A \uparrow \Rightarrow v \downarrow$
 $A \uparrow \Rightarrow p \downarrow$
 $A \uparrow \Rightarrow v \uparrow$
- dp, dv always opposite signs
($v \uparrow \Rightarrow T \downarrow \Rightarrow p \downarrow$)



Isentropic Flow with Area Change -3

Copyright © 2001-2002 by Jerry M. Setzler. All rights reserved.

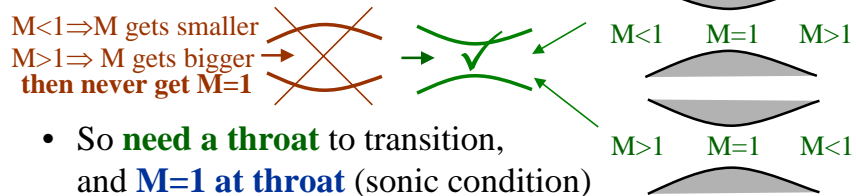
AE3450

Sonic Throat Requirement

$$\frac{dA}{A} = \frac{dp}{\rho v^2} (1 - M^2) = -\frac{dv}{v} (1 - M^2)$$

• How to transition from subsonic to supersonic (or vica versa)?

- Need to go through $M=1$
- For $M=1: dA$ or $dv = 0$
but $dv \neq 0$ (flow accel. or decel.)
- $dA=0$: maximum or minimum in area



Isentropic Flow with Area Change -4

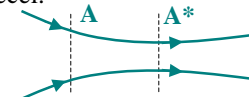
Copyright © 2001-2002 by Jerry M. Setzler. All rights reserved.

AE3450

Area Ratio

- For isentropic flow, look at effect of area change on M by comparing A at any point to area at sonic point (A*)

- * refers to properties of a flow if isen. accel./decel. to M=1 sonic conditions (e.g., ρ*, T*,...)
- alternative to stagnation as ref. state



- Use mass conservation to find relation

$$\rho v A = \rho^* v^* A^* \quad \frac{1}{M} \quad \frac{\rho_o}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{1/\gamma-1} \text{ from VI.8}$$

$$\frac{A}{A^*} = \frac{\rho^* v^*}{\rho v} = \frac{\rho^* \rho_o}{\rho \rho_o} \frac{v^*}{v} \frac{a^*}{a} \quad \frac{\rho_o}{\rho^*} = \left(1 + \frac{\gamma-1}{2}\right)^{1/\gamma-1}$$

$$\frac{A}{A^*} = \frac{1}{M} \left\{ \frac{1 + \frac{\gamma-1}{2} M^2}{\frac{\gamma+1}{2}} \right\}^{\gamma+1/2(\gamma-1)} \quad \frac{a^*}{a} = \sqrt{\frac{T^*}{T}} = \sqrt{\frac{T^* T_o}{T T_o}} \text{ with } \frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2$$

$$\frac{T_o}{T^*} = 1 + \frac{\gamma-1}{2} \quad \text{from VI.6}$$

(VI.17) for TPG, CPG

Isentropic Flow with Area Change © Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

Area Ratio Results

- Two (isentropic) solutions for a given A/A*

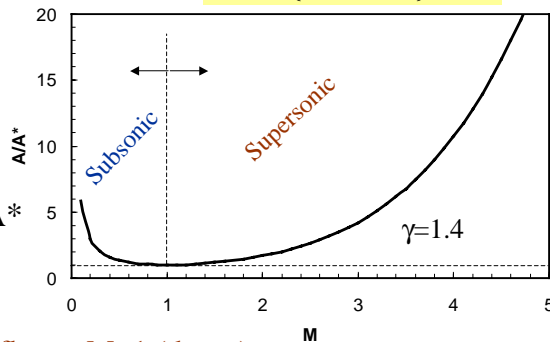
- one subsonic
- one supersonic

- A ≥ A* always
- Accel. to high M requires large A/A*

$$\frac{\dot{m}/A}{(\dot{m}/A)^*} = \frac{\rho v}{\rho^* v^*} = \frac{A^*}{A}$$

- maximum mass flux at M=1 (throat)

$$\frac{A}{A^*} = \frac{1}{M} \left\{ \frac{1 + \frac{\gamma-1}{2} M^2}{\frac{\gamma+1}{2}} \right\}^{\gamma+1/2(\gamma-1)}$$



Isentropic Flow with Area Change © Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

Mass Flux and Stagnation Properties

- Examine mass flux in terms of stagnation conditions

$$\frac{\dot{m}}{A} = \rho v = \frac{p}{RT} Ma = \frac{p}{RT} M \sqrt{\gamma RT} \quad \text{from VI.2, TPG}$$

$$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}} \frac{\sqrt{\gamma} M}{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma+1/2(\gamma-1)}} \quad \text{from VI.6,7 CPG}$$

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2$$

$$\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma/\gamma-1} \quad \text{(VI.18)}$$

$$\text{or } \frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}} f(\gamma, M)$$

- For given isen. flow, all stagnation (and sonic) properties constant, including mass flow rate

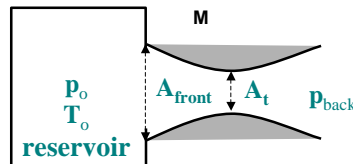
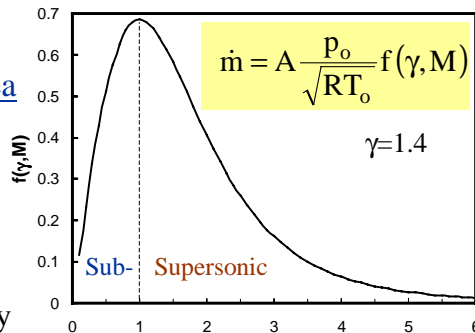
Isentropic Flow with Area Change - 7

Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

Choked Flow

- For fixed stagnation properties and flow area
 - max \dot{m} at $M=1$
- For nozzle with fixed stagnation properties
 - if throat is sonic, can't get more \dot{m} by changing downstream conditions (e.g., back pressure)
 - \Rightarrow **Choked Flow**



Isentropic Flow with Area Change - 8

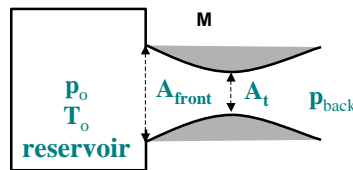
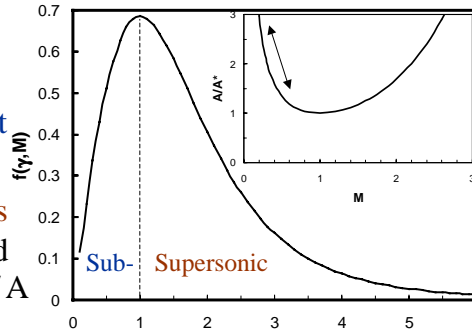
Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

Choked Flow (con't)

- For nozzle with **fixed stagnation properties** and **initially sonic throat**

- if **reduce throat area** A_t , flow at **throat stays sonic** ($A_{front}/A_t \uparrow$) and $\dot{m} \downarrow$ (since $A \downarrow$, & \dot{m}/A same)
- if **increase** A_t , $\dot{m} \uparrow$ and eventually throat not sonic ($A_{front}/A^* \downarrow$) (not choked and $\dot{m} \neq \dot{m}_{max}$)



Isentropic Flow with Area Change -9

Copyright © 2001-2002 by Jerry M. Setzler. All rights reserved.

AE3450

Choked Mass Flowrate

- Maximum flow rate when choked ($M=1$ at throat)
- Choked mass flowrate from $f(\gamma, 1)$

$$\dot{m}_{max} = A^* \frac{P_o}{\sqrt{RT_o}} \sqrt{\gamma} \left(1 + \frac{\gamma-1}{2} \right)^{\gamma+1/2(1-\gamma)} \quad (VI.19)$$

- To **increase mass flowrate**
 - **increase** A^* (throat size)
 - increase p_o , decrease T_o (**increase stagnation density**)
- $f(\gamma, 1)$ typically near 0.7

$$\dot{m}_{max} \approx 0.7 \frac{P_o}{\sqrt{RT_o}} A_{throat} \quad f(\gamma, 1) = \begin{cases} 0.726 & \gamma = 5/3 \\ 0.685 & \gamma = 1.4 \\ 0.667 & \gamma = 1.3 \end{cases}$$

rule of thumb for **choked** gas flows

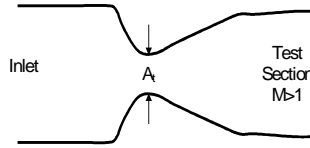
Isentropic Flow with Area Change -10

Copyright © 2001-2002 by Jerry M. Setzler. All rights reserved.

AE3450

Example 1

- For a supersonic wind tunnel with an incoming flow with $M \ll 1$ (from a high pressure reservoir or compressor), need a throat to produce $M > 1$
- What area throat required to produce a test section Mach number of $M=3$ in test section with 0.2 m^2 cross-section?
 - Assume isentropic flow, calor./thermally perfect gas, $\gamma=1.4$.



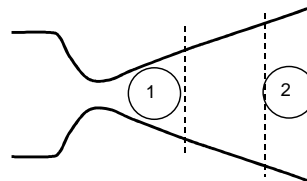
Isentropic Flow with Area Change -11

Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

Example 2

- Converging-diverging supersonic nozzle with $M_1=2.0$ and $M_2=3.0$
- What is A_2/A_1 ?
 - Assume isentropic flow, calor./thermally perfect gas, $\gamma=1.4$.



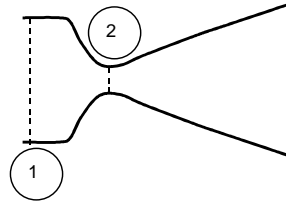
Isentropic Flow with Area Change -12

Copyright © 2001-2002 by Jerry M. Setzman. All rights reserved.

AE3450

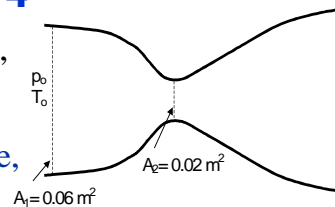
Example 3

- For nozzle shown, inlet area $A_1=0.50 \text{ m}^2$, $M_1=0.20$ and p_o , T_o fixed
- How far can A_2 be reduced from A_1 without changing mass flowrate in nozzle?
 - Assume isentropic flow, calor./thermally perfect gas, $\gamma=1.4$.



Example 4

- For nozzle shown ($A_1=0.060\text{m}^2$, $A_2=0.020\text{m}^2$) and p_o , T_o fixed
- For isentropic flow in the nozzle, what are the limits on the allowed inlet Mach numbers (M_1)?
 - Assume tpg/cpg, $\gamma=1.4$.



Example 5

- Same nozzle as Example 4, air and $p_o=500\text{kPa}$, $T_o=300\text{K}$
- What is max. possible mass flowrate through the nozzle?
 - Assume $\gamma=1.4$.

