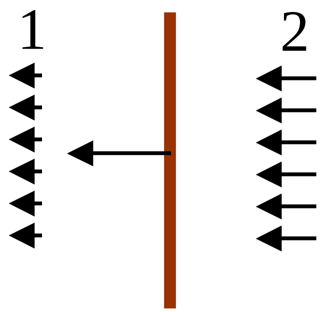
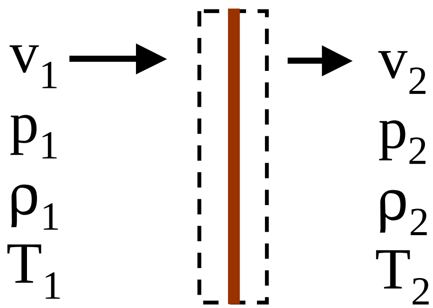
Normal Shock Waves

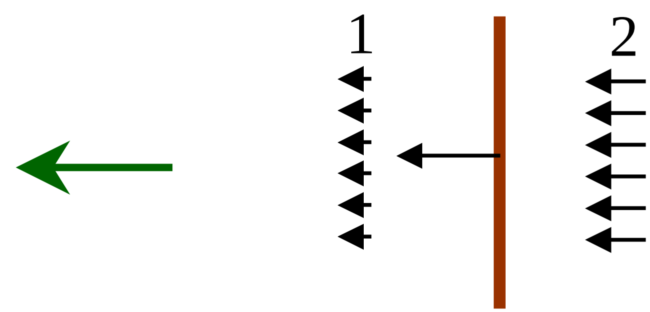


- Wave propagation is perpendicular to flow direction
- Shock is nonequilibrium process internally, but
  - flow *before shock* (1) in equilibrium
  - flow *after shock* (2) in equilibrium

Approach to Finding Shock Properties

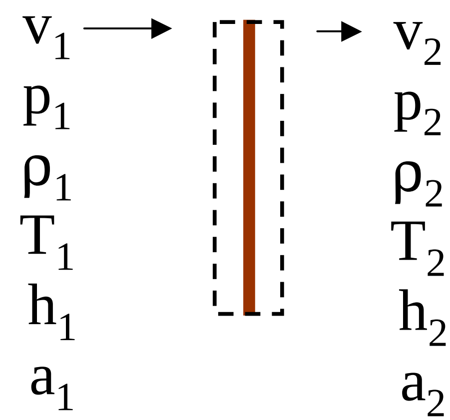
- Start with stationary shock
  ⇒ can use **steady** equations





- Use **control volume** analysis
  – only need to consider properties before and after shock (equilibrium)
- Equations first studied by Rankine (~1870) and Hugoniot (~1877)

Governing Equations



- Conservation and state equations
  - 1d, steady, inviscid except inside shock adiabatic, only flow work

Mass

$$\frac{\dot{m}}{A} = \rho_1 v_1 = \rho_2 v_2 \qquad \textbf{(VII.1)}$$

Momentum

$$p_1 A - p_2 A = \dot{m}_2 v_2 - \dot{m}_1 v_1 \rightarrow p_1 - p_2 = \frac{\dot{m}}{A}\left(v_2 - v_1\right) \qquad \textbf{(VII.2a)}$$

$$p_1 A - p_2 A = \rho_2 v_2^2 A - \rho_1 v_1^2 A \rightarrow p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \qquad \textbf{(VII.2b)}$$

Energy

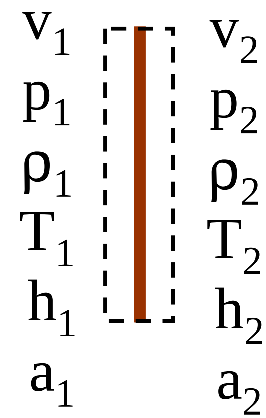
$$h_1 + v_1^2/2 = h_2 + v_2^2/2 = h_o \qquad \textbf{(VII.3)}$$

Perfect Gas State Eqns.

$$p = \rho R T \ \textbf{(a)} \qquad dh = c_p dT \ \textbf{(b)} \qquad a^2 = \gamma R T \ \text{(VI.2)}$$

- **6 equations, 6 unknowns**

Shock Density Ratio



- Combine energy, momentum, mass

(VII.3) → $$h_2 - h_1 = \frac{1}{2}\left(v_1^2 - v_2^2\right) = \frac{1}{2}\left(v_1 - v_2\right)\left(v_1 + v_2\right)$$

(VII.2a) $$\frac{p_1 - p_2}{\dot{m}/A} = \left(v_2 - v_1\right)$$

$$\frac{\dot{m}}{A}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = v_1 + v_2 \quad \text{(VII.1)}$$

Shock Hugoniot Eq.

(VII.4) $$h_2 - h_1 = \frac{1}{2}\left(p_2 - p_1\right)\left(1/\rho_1 + 1/\rho_2\right)$$

- True for all simple comp. substances

TPG/CPG

(b) (a)

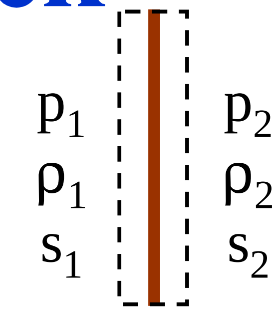
$$h_2 - h_1 = c_p\left(T_2 - T_1\right) = \frac{R\gamma}{\gamma - 1}\left(p_2/R\rho_2 - p_1/R\rho_1\right)$$

- Only functions of p and ρ
- Solve for density ratio →

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\gamma+1}{\gamma-1}\dfrac{p_2}{p_1}}{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}}$$

(VII.5)

Entropy Change Across Shock



- For **TPG/CPG**, entropy state equation

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1}$$

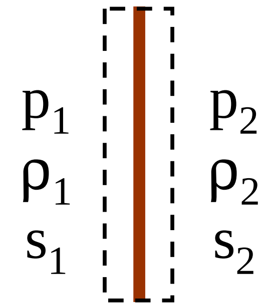
$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{T_2}{T_1}\left(\frac{\rho_2}{\rho_1}\right)^{-(\gamma-1)}\right] = \ln\left[\frac{p_2/\rho_2}{p_1/\rho_1}\left(\frac{\rho_2}{\rho_1}\right)^{-(\gamma-1)}\right]$$

$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right] \quad \textbf{(VII.6)}$$

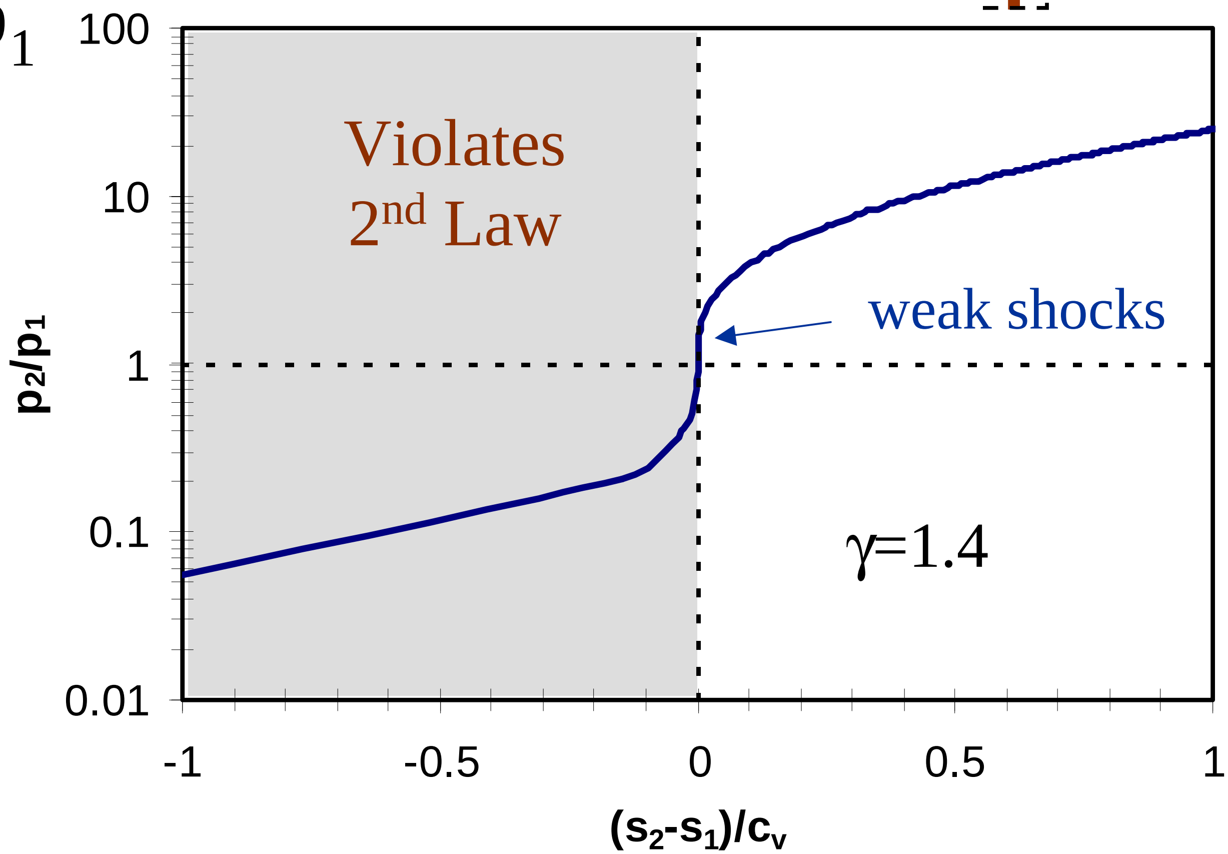
- **Entropy change as function of pressure and density ratios across shock**

Entropy Change Across Shock

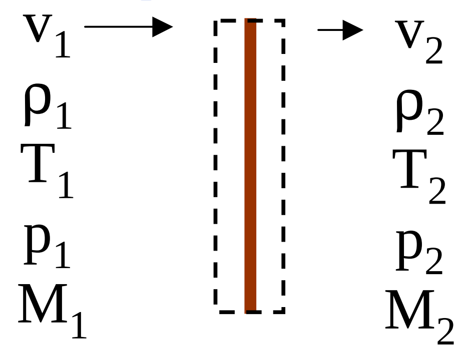
$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right]$$



- Combine with ρ_2/ρ_1 from (VI.25)
- $p_2 < p_1$ violates Second Law
  - – no expansion shocks



Mach Number Relations: v, ρ



- General problem is to find change in all properties across shock
- Start by find changes based on M_1, M_2
  - then find M_2 as function of M_1
- **Velocity ratio,** v_2/v_1

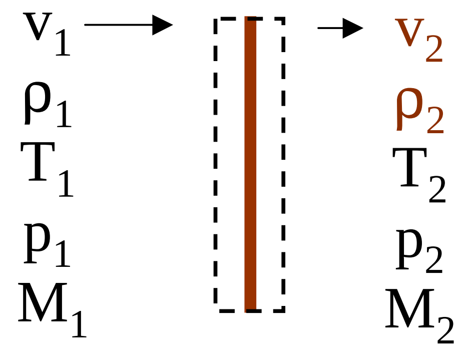
$$\frac{v_2}{v_1} = \frac{M_2 a_2}{M_1 a_1} \rightarrow \frac{v_2}{v_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \qquad \textbf{(VII.7)}$$

- **Density ratio,** ρ_2/ρ_1

mass $\quad \rho_1 v_1 = \rho_2 v_2 \rightarrow \dfrac{\rho_2}{\rho_1} = \dfrac{v_1}{v_2} = \dfrac{M_1}{M_2}\sqrt{\dfrac{T_1}{T_2}} \qquad$ **(VII.8)**

Mach Number Relations: T

- **Temperature ratio,** T_2/T_1



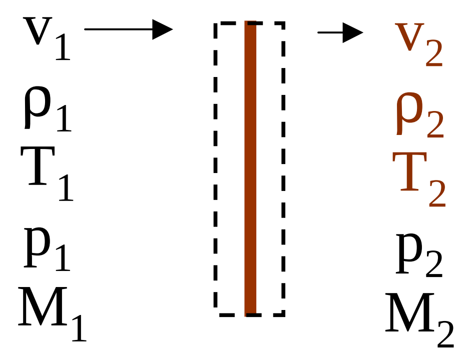
energy $\quad h_{o1} = h_{o2} \Rightarrow T_{o1} = T_{o2}$

$$\frac{T_2}{T_1} = \frac{T_2/T_{o2}}{T_1/T_{o1}} \frac{T_{o2}}{T_{o1}} \quad \left(\frac{T_{o2}}{T_{o1}} = 1\right)$$

$$= \frac{1\Big/\left(1+\frac{\gamma-1}{2}M_2{}^2\right)}{1\Big/\left(1+\frac{\gamma-1}{2}M_1{}^2\right)} \rightarrow \frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)} \quad \textbf{(VII.9)}$$

Mach Number Relations: p

- **Pressure ratio,** p_2/p_1



momentum
(VII.2b)

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2$$

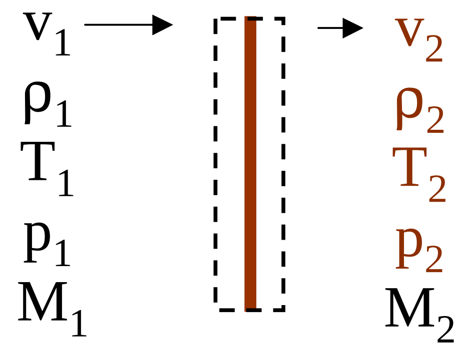
$$\rho v^2 = \rho\left(M^2 a^2\right)$$
$$= \rho\left(M^2 \gamma R T\right)$$
$$= \rho\left(M^2 \gamma \frac{p}{\rho}\right)$$
$$= p \gamma M^2$$

$$p_1 + p_1 \gamma M_1^2 = p_2 + p_2 \gamma M_2^2$$

$$p_1\left(1 + \gamma M_1^2\right) = p_2\left(1 + \gamma M_2^2\right)$$

$$\frac{p_2}{p_1} = \frac{\left(1 + \gamma M_1^2\right)}{\left(1 + \gamma M_2^2\right)} \qquad \textbf{(VII.10)}$$

Mach Number Relations: M



- **Mach Number,** $M_2 = f(M_1)$
  - combine all eqn's.

mass (VII.1)

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1}$$

M and a (VI.2,3)

$$\frac{M_1\sqrt{\gamma R T_1}}{M_2\sqrt{\gamma R T_2}} = \frac{v_1}{v_2}$$

PG (a)

$$\frac{\rho_2}{\rho_1} = \frac{p_2/RT_2}{p_1/RT_1}$$

mom. (VII.9)

$$\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)} = \frac{p_2}{p_1}$$

energy (VII.10)

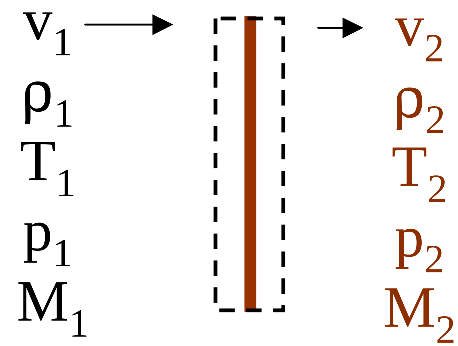
$$\frac{T_2}{T_1} = \left(1+\frac{\gamma-1}{2}M_1^{\,2}\right) \Big/ \left(1+\frac{\gamma-1}{2}M_2^{\,2}\right)$$

$$\frac{M_2}{1+\gamma M_2^2}\sqrt{1+\frac{\gamma-1}{2}M_2^2} = \frac{M_1}{1+\gamma M_1^2}\sqrt{1+\frac{\gamma-1}{2}M_1^2}$$

- Expression of M_2 as function of M_1 - **solve**

Mach Number Relations: M (con't)

- Remove square roots by squaring both sides and solve resulting quadratic



$$\frac{M_2^2}{\left(1+\gamma M_2^2\right)^2}\left(1+\frac{\gamma-1}{2}M_2^2\right)=\underbrace{\frac{M_1^2}{\left(1+\gamma M_1^2\right)^2}\left(1+\frac{\gamma-1}{2}M_1^2\right)}_{\equiv g(M_1)}$$

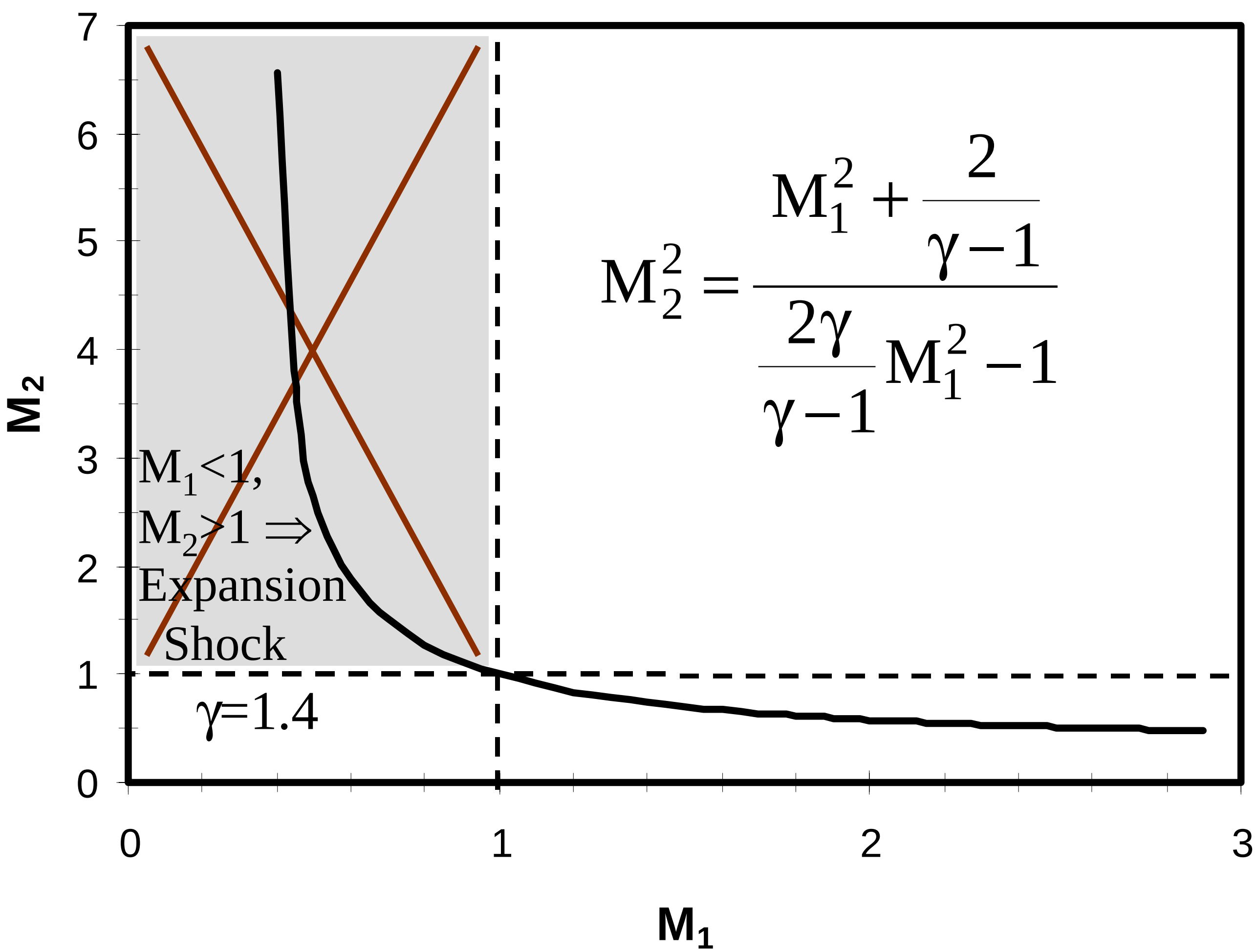
$$M_2^4\left[\frac{\gamma-1}{2}-\gamma^2 g(M_1)\right]+M_2^2\left[1-2\gamma g(M_1)\right]-g(M_1)=0$$

$$M_2^2=\frac{M_1^2+\dfrac{2}{\gamma-1}}{\dfrac{2\gamma}{\gamma-1}M_1^2-1} \quad \text{or} \quad M_1^2$$

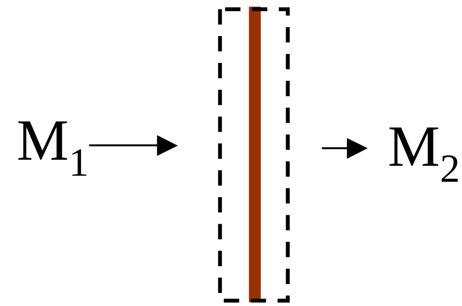
"No shock" solution

(VII.11)

Mach Change Across Normal Shock

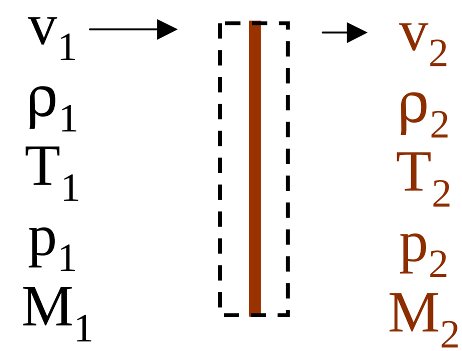


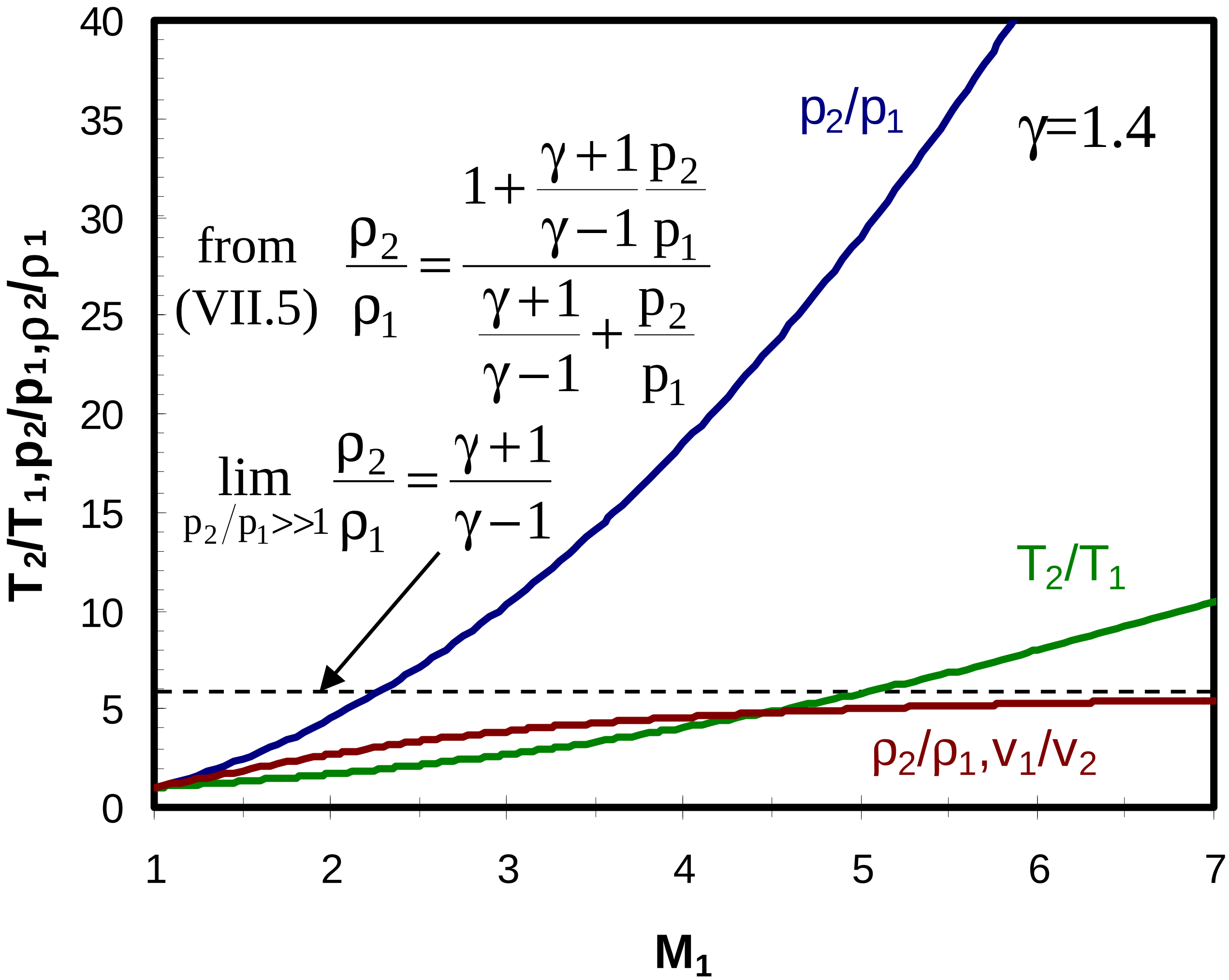
$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}$$



- In *reference frame* of normal shock, flow **after** shock is always **subsonic**

Property Ratios - Results





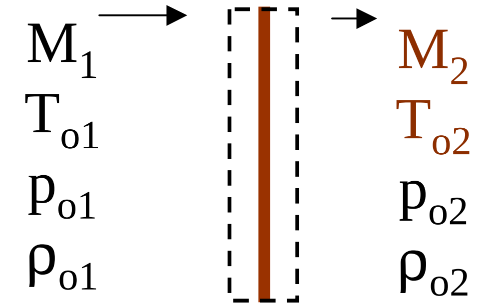
from (VII.5) $$\frac{\rho_2}{\rho_1} = \frac{1+\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}}$$

$$\lim_{p_2/p_1 >> 1} \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$$

- T, p and ρ **increase**, v **decreases**
- p increase across normal shock is **greatest static property change**
- Density ratio and velocity ratio approach **limit**

Stagnation Properties Across Shock

- **Stagnation Temperature,** $T_{o2}=T_{o1}$
- **Stagnation Pressure**



$$\frac{p_{o2}}{p_{o1}}=\frac{p_{o2}/p_2}{p_{o1}/p_1}\frac{p_2}{p_1}=\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}$$

from (VII.10)

$$\frac{p_2}{p_1}=\frac{\left(1+\gamma M_1^2\right)}{\left(1+\gamma M_2^2\right)}$$

M_2 from (VII.11)

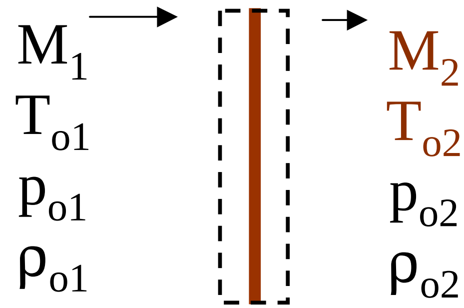
$$\frac{p_2}{p_1}=\frac{2\gamma}{\gamma+1}M_1^2-\frac{\gamma-1}{\gamma+1} \qquad \textbf{(VII.12)}$$

$$\frac{p_{o2}}{p_{o1}}=\left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2-\frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}$$

(VII.13)

Stagnation Pressure

- **Other Useful Expressions**



$$\frac{p_{o2}}{p_{o1}} = \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2} \right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1}$$

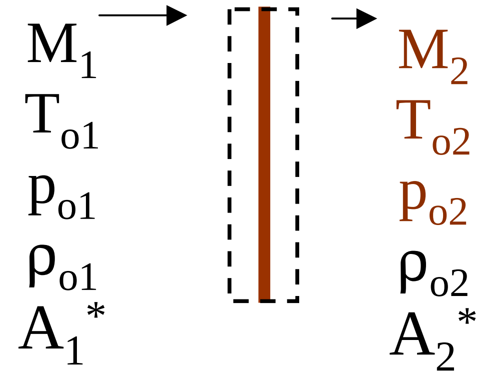
$$\frac{p_{o2}}{p_{o1}} = \left(\frac{T_1}{T_2} \right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1} \qquad \text{(VII.14)}$$

$$\frac{p_{o2}}{p_1} = \frac{p_{o2}}{p_2}\frac{p_2}{p_1}$$

$$\frac{p_{o2}}{p_1} = \left(1+\frac{\gamma-1}{2}M_2^2 \right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1} \qquad \text{(VII.15)}$$

Stagnation Properties (con't)

- **Stagnation Density,** ρ_{o2}/ρ_{o1}



P.G.

$$\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}} \Big/ \frac{RT_{o2}}{RT_{o1}}$$

1

$$\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}} \quad \textbf{(VII.16)}$$

- **Sonic Area Ratio,** A_2^*/A_1^*

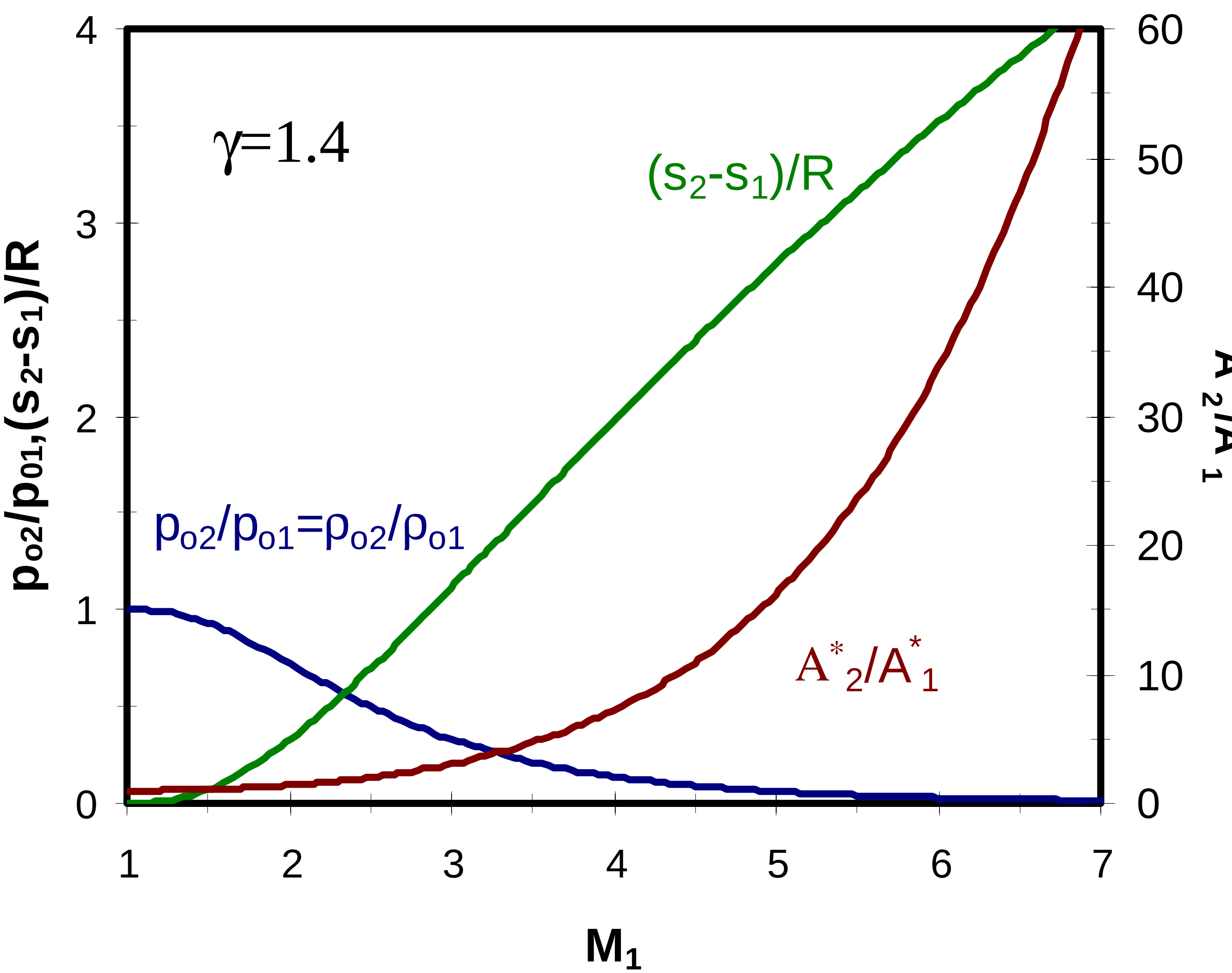
from (VI.19)

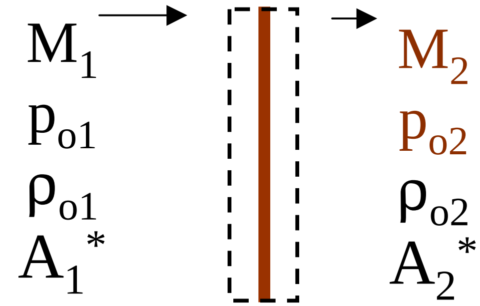
$$\frac{\dot{m}_2}{\dot{m}_1} = \frac{A_2^*}{A_1^*} \frac{p_{o2}\big/\sqrt{RT_{o2}}}{p_{o1}\big/\sqrt{RT_{o1}}} \frac{f(\gamma)}{f(\gamma)}$$

1 1 1

$$\Rightarrow \frac{A_2^*}{A_1^*} = \frac{p_{o1}}{p_{o2}} \quad \textbf{(VII.17)}$$

Stag. Press., Entropy and A* - Results

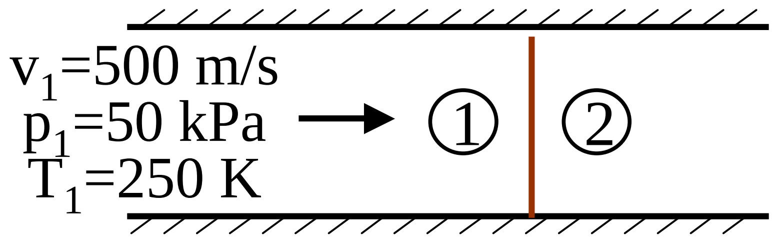




- p_o , ρ_o across normal shock **drop** (alot)
- Entropy **increases**
- Sonic area ratio **increases**
  - –**larger throat required after shock** to reach sonic flow (*same mass flowrate, less* p_o) $\dot{m} \propto A^* p_o$

Examples: Problem 1

- **Given**: Air flowing through constant area duct encounters stationary shock
  - oncoming air 500 m/s, 50 kPa, 250K



- **Find**:

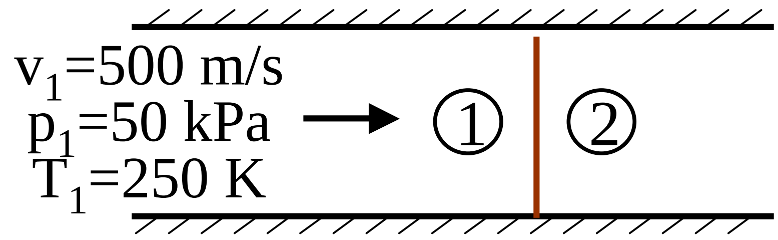
  $\mathbf{M_2, T_2, p_2, v_2}$

  $\mathbf{p_{o2}}$, $\mathbf{T_{o2}}$ (relative to shock/duct)

- **Assume**: Air is TPG, CPG, γ=1.4

Solution: Problem 1

- **Analysis:**
  - first calculate M_1



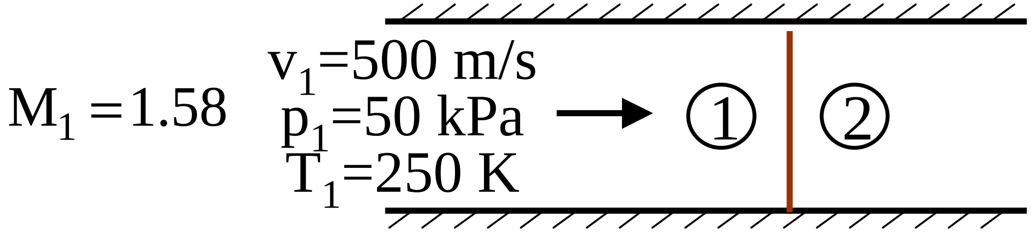
$$M_1 = \frac{v_1}{\sqrt{\gamma RT}} = \frac{500\,m/s}{20\sqrt{250}\,m/s} = 1.58$$

  - can either use equations or tables (e.g., Table B.1 for γ=1.4)

M_1	M_2	p_2/p_1	T_2/T_1	p_{o2}/p_{o1}	ρ_2/ρ_1	A^*_2/A^*_1	p_{o2}/p_1
1.50	0.7011	2.458	1.320	0.9298	1.862	1.076	3.413
1.51	0.6976	2.493	1.327	0.9266	1.879	1.079	3.451
1.52	0.6941	2.529	1.334	0.9233	1.896	1.083	3.489
1.53	0.6907	2.564	1.340	0.9200	1.913	1.087	3.528
1.54	0.6874	2.600	1.347	0.9166	1.930	1.091	3.567
1.55	0.6841	2.636	1.354	0.9132	1.947	1.095	3.606
1.56	0.6809	2.673	1.361	0.9097	1.964	1.099	3.645
1.57	0.6777	2.709	1.367	0.9062	1.981	1.104	3.685
1.58	0.6746	2.746	1.374	0.9026	1.998	1.108	3.724
1.59	0.6715	2.783	1.381	0.8989	2.015	1.112	3.765

Solution: Problem 1 (con't.)

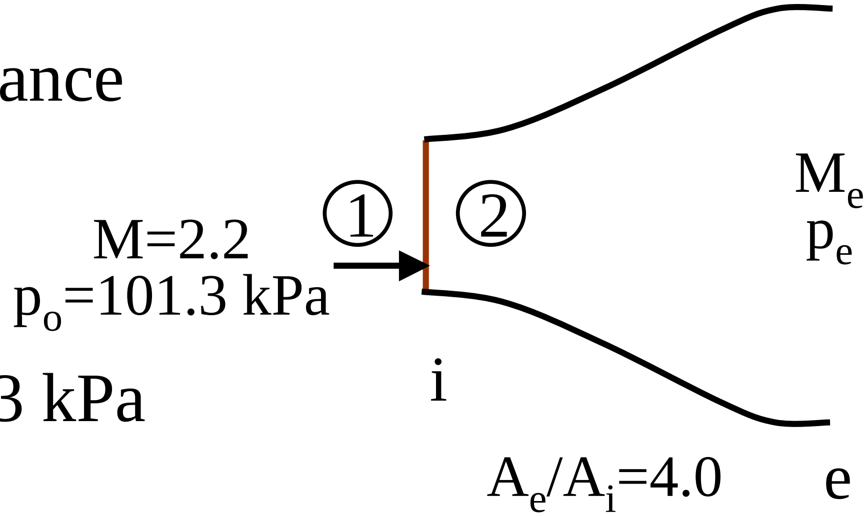
- **Analysis (con't):**



- M_2: B.1/VII.11 $M_2^2 = \left(1.58^2 + \frac{2}{1.4-1}\right) \Big/ \left(\frac{2(1.4)}{1.4-1} 1.58^2 - 1\right)$ **M_2=0.675**
- T_2: B.1/VII.9 $\frac{T_2}{T_1} = \left(1 + \frac{1.4-1}{2} 1.58^2\right) \Big/ \left(1 + \frac{1.4-1}{2} 0.675^2\right) = 1.374$ **T_2=344 K**
- p_2: B.1/VII.12 $p_2/p_1 = [(2 \times 1.4)/0.4] 1.58^2 - [0.4/(1.4+1)] = 2.746$ **p_2=137 kPa**
- v_2: B.1/VII.7 $v_1/v_2 = \rho_2/\rho_1 = (1.58/0.675)\sqrt{1/1.374} = 1.998$ **v_2=250 m/s**
- p_{o2}: A.1/VII.15 $p_{o2}/p_1 = \left(1 + \frac{1.4-1}{2} 0.675^2\right)^{3.5} 2.746 = 3.724$ **p_{o2}=186 kPa**
- T_{o2}: A.1/VI.6 $T_{o2} = T_{o1} = 250\text{K}\left(1 + \frac{1.4-1}{2} 1.58^2\right) = (250/0.667)\text{K}$ **T_{o2}=375 K**

Examples: Problem 2

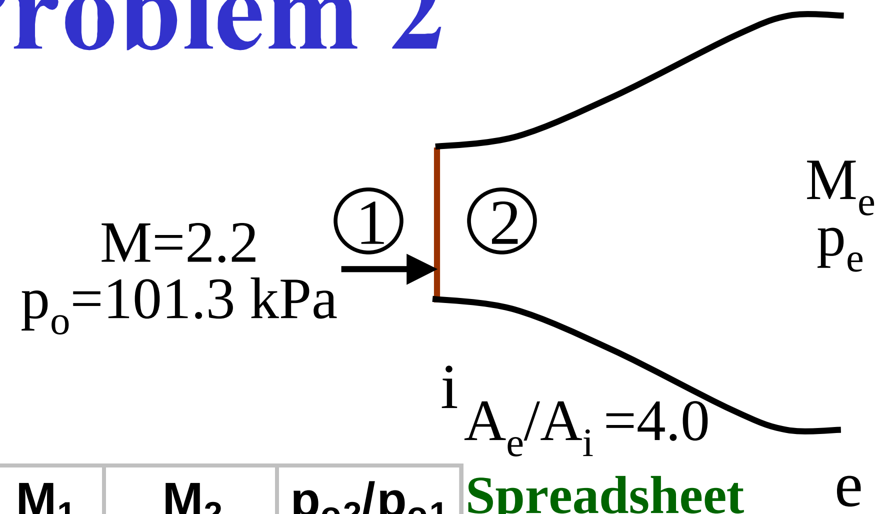
- **Given:** Expanding nozzle with exit/inlet area ratio of 4.0
  - normal shock stands at entrance
  - just before entrance He, Mach 2.2 with stagnation pressure of 101.3 kPa



- **Find:**
  1. Mach number at nozzle exit
  2. (static) pressure at exit

- **Assume:** He is TPG/CPG with γ=5/3

Solution: Problem 2



- **Analysis:**
  - **M_e**: get M_e from area ratio

isentropic 2→e

$$\frac{A_e}{A_i} = \frac{\left(A_e/A^*\right)_{M_e}}{\left(A_i/A^*\right)_{M_2}} = 4.0$$

  - but need M_2 **VII.11/B.3**

$$M_1 = M_i = 2.2 \quad \Rightarrow \quad M_2 = 0.581$$

M_1	M_2	p_{o2}/p_{o1}
2.20	0.5813	0.6860

Spreadsheet of VII.11,13

$$\left.\frac{A_e}{A^*}\right|_{M_e} = 4\left.\frac{A_i}{A^*}\right|_{0.581} = 4\times1.20 = 4.80 \Rightarrow M_e = \begin{cases} 0.118 \\ 3.73 \end{cases}$$

VI.17/A.3

VI.17,7

M	A/A*	p/p_o
0.118	4.799	0.9884
0.581	1.198	0.0906
3.732	4.799	0.0132

  - **p_e:**

$$p_e = p_{oe}\frac{p_e}{p_{oe}} = \left(\frac{p_{o2}}{p_{o1}}p_{o1}\right)\left.\frac{p}{p_o}\right|_{M_e=0.118}$$

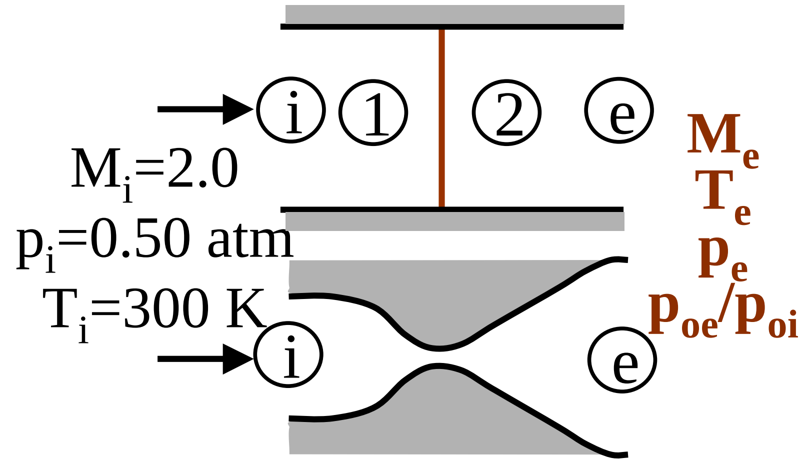
VII.13/B.3 **VI.7/A.3**

$$p_e = \left(0.686\times101.3\text{kPa}\right)0.988$$

p_e=68.7 kPa

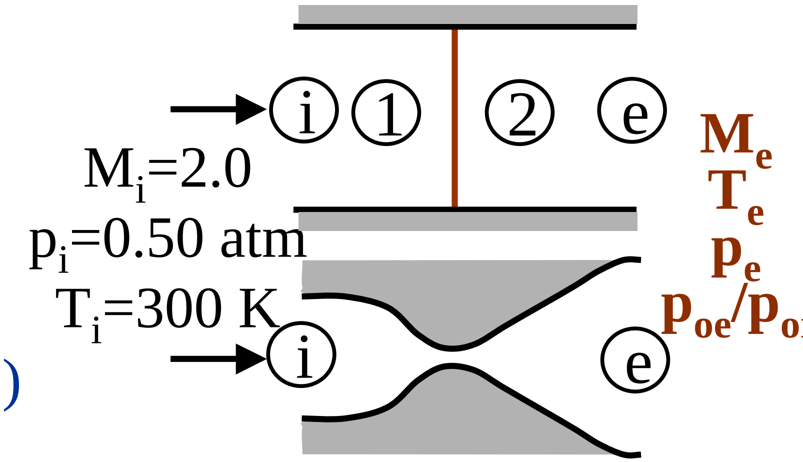
Examples: Problem 3

- **Given:** Two engine inlets, one a straight tube, the other a converging-diverging diffuser.
  - both with M=2.0 at their inlets, T_i=300.K, p_i=0.50 atm
  - both slow flow down to same M_e
    CD diffuser: isentropically
    straight diffuser: with shock



- **Compare performance of diffusers:** **M_e, T_e, p_e, p_{oe}/p_{oi}**
- **Assumptions:** air is cpg/tpg with γ=1.4
  adiabatic diffusers
  isentropic except at shock

Solution: Problem 3



- **Analysis: "Shock Diffuser"**
  - γ=1.4, VII.8-11, 13, 17 or

M_1	M_2	p_2/p_1	T_2/T_1	ρ_2/ρ_1	p_{o2}/p_{o1}	A^*_2/A^*_1
2.00	0.5774	4.500	1.688	2.667	0.7209	1.387

  - $\mathbf{M_e}=M_2=\mathbf{0.577}$ (spreadsheet like B.1)
  - $\mathbf{T_e}=T_2=(T_2/T_1)T_1=(1.688)300K=$ **506 K**
  - $\mathbf{p_e}=p_2=(p_2/p_1)p_1=(4.50)0.5atm=$**2.25 atm**; $\mathbf{p_{oe}/p_{oi}}=p_{o2}/p_{o1}=$ **0.721**
- **"CD Diffuser"** - isentropic (VI.6, 7, 17 or..) (spreadsheet like A.1)

M	A/A*	T/T_o	p/p_o
0.577	1.217	0.9376	0.7980
2.000	1.688	0.5556	0.1278

  - $\mathbf{M_e}=\mathbf{0.577}$
  - $\mathbf{T_e}=\frac{(T_e/T_o)|_{0.577}}{(T_i/T_o)|_{2.0}}T_i=\frac{0.938}{0.556}\times 300K=$ **506 K** — same (M, T_o const)
  - $\mathbf{p_e}=(T_e/T_i)^{\gamma/\gamma-1}p_i=(506/300)^{3.5}0.5atm=$**3.12 atm** ← higher p
  - $\mathbf{p_{oe}/p_{oi}}=\mathbf{1}$ ← no p_o loss ⇒ higher mass flowrate or smaller area