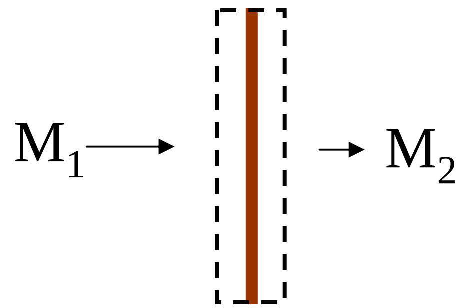
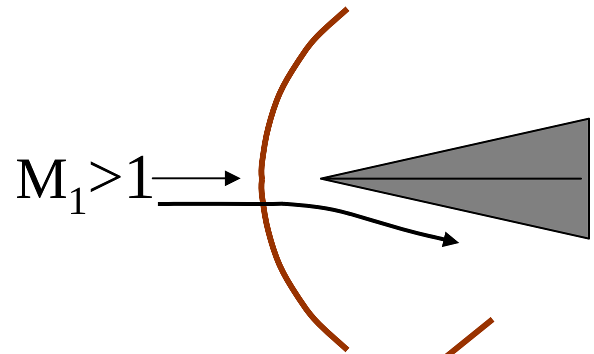
Supersonic Flow Turning

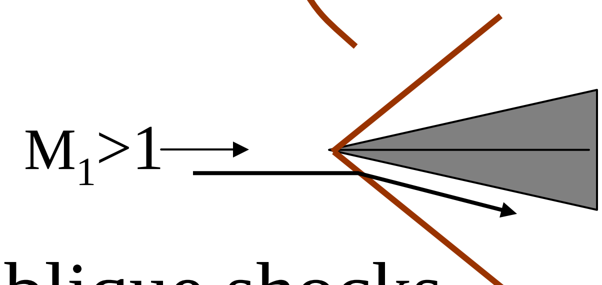
- For normal shocks, flow is perpendicular to shock
  - no change in flow direction



- How does supersonic flow change direction, i.e., make a turn
  - either slow to subsonic ahead of turn (can then make gradual turn) **=bow shock**



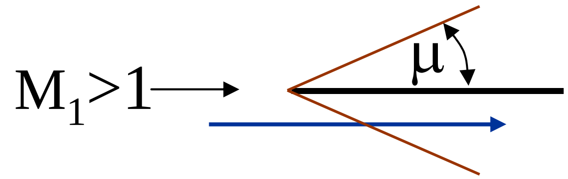
  - go through non-normal wave with sudden angle change , i.e., **oblique shock** (also expansions: see later)



- Can have straight/curved, 2-d/3-d oblique shocks
  - will examine straight, 2-d oblique shocks

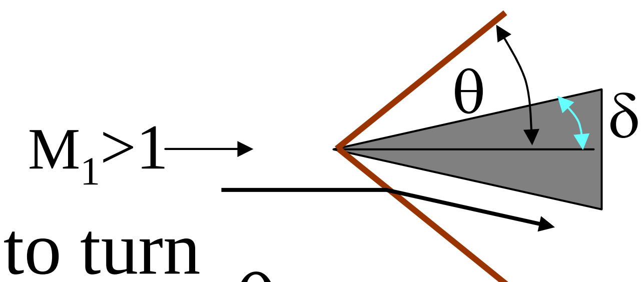
Oblique Shock Waves

- Recall Mach wave
  - consider infinitely thin body
  - no flow turn required
  - infinitessimal wave

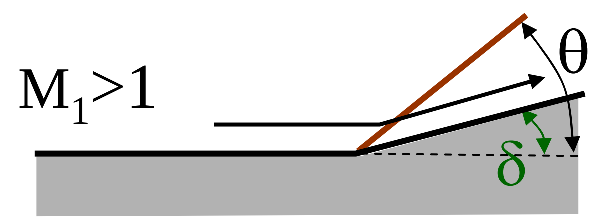


$$\mu = \sin^{-1}(1/M_1)$$

- Oblique shock
  - consider finite-sized wedge, half-angle, δ
  - flow must undergo compression to turn
  - if attached shock
    $\Rightarrow$ oblique shock at angle θ
  - similar for concave corner



$$\theta > \mu$$

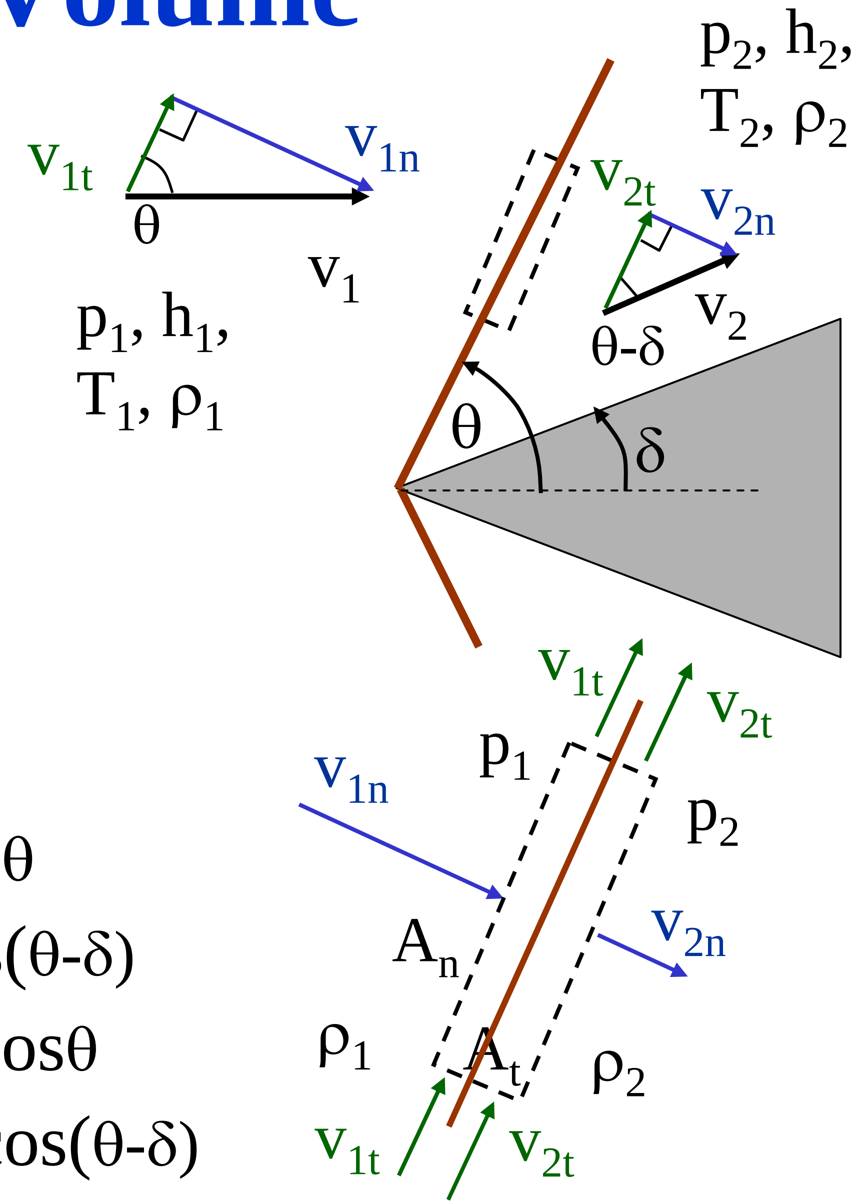


Equations of Motion

- Governing equations
  - same approach as for normal shocks
  - use conservation equations and state equations
- Conservation Equations
  - mass, energy and momentum
  - this time 2 momentum equations - 2 velocity components for a 2-d oblique shock
- Assumptions
  - steady flow (stationary shock), inviscid except inside shock, adiabatic, no work but flow work

Control Volume

- Pick control volume along shock
- Divide velocity into two components
  - one tangent to shock, $\mathbf{v_t}$
  - one normal to shock, $\mathbf{v_n}$
- Angles from geometry
  - $v_{1n}=v_1\sin\theta$; $v_{1t}=v_1\cos\theta$
  - $v_{2n}=v_2\sin(\theta\text{-}\delta)$; $v_{2t}=v_2\cos(\theta\text{-}\delta)$
  - $M_{1n}=M_1\sin\theta$; $M_{1t}=M_1\cos\theta$
  - $M_{2n}=M_2\sin(\theta\text{-}\delta)$; $M_{2t}=M_2\cos(\theta\text{-}\delta)$

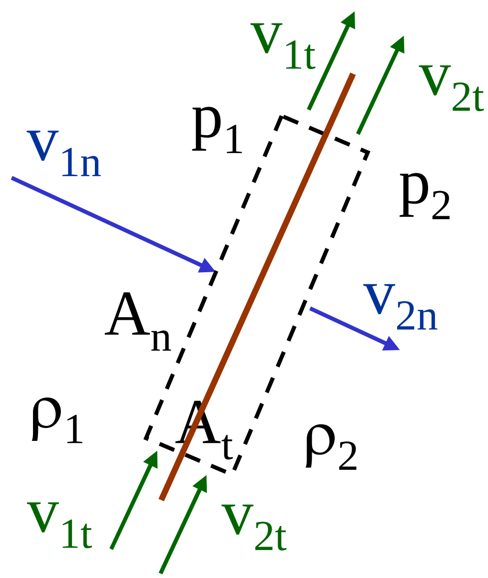


Conservation Equations

- **Mass** $$0 = \int_{CS} \rho\left(\vec{v}_{rel} \cdot \vec{n}\right) dA$$

$$\rho_1 v_{1n} A_n + \cancel{\rho_1 v_{1t} \frac{A_t}{2}} + \cancel{\rho_2 v_{2t} \frac{A_t}{2}} = \rho_2 v_{2n} A_n + \cancel{\rho_1 v_{1t} \frac{A_t}{2}} + \cancel{\rho_2 v_{2t} \frac{A_t}{2}}$$

$$\boxed{\rho_1 v_{1n} = \rho_2 v_{2n}} \qquad \textbf{(1)}$$



- **Momentum** $$-\int_{CS} p \cdot \vec{n}\, dA = \int_{CS} \rho \vec{v}\left(\vec{v}_{rel} \cdot \vec{n}\right) dA$$

tangent $$\cancel{(p_1 + p_2)\frac{A_t}{2} - (p_1 + p_2)\frac{A_t}{2}} = v_{1t}\left(-\cancel{\rho_1 v_{1n} A_n}\right) + v_{2t}\left(\cancel{\rho_2 v_{2n} A_n}\right)$$

$$\boxed{v_{1t} = v_{2t}}$$

normal $$p_1 A_n - p_2 A_n = v_{1n}\left(-\rho_1 v_{1n} A_n\right) + v_{2n}\left(\rho_2 v_{2n} A_n\right)$$

$$\boxed{p_1 - p_2 = \rho_1 v_{1n}^2 + \rho_2 v_{2n}^2} \qquad \textbf{(2)}$$

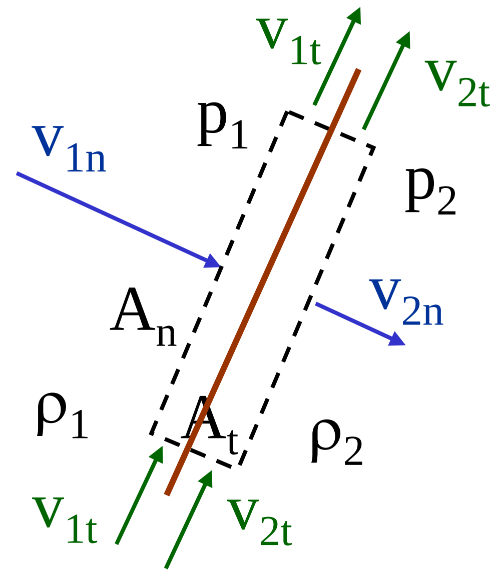
Conservation Equations (con't)

- **Energy**

$$\cancel{\rho_1 v_{1n}} A_n\left(h_1+\frac{v_1^2}{2}\right)=\cancel{\rho_2 v_{2n}} A_n\left(h_2+\frac{v_2^2}{2}\right)$$

$$h_1+\frac{v_{1n}^2}{2}+\cancel{\frac{v_{1t}^2}{2}}=h_2+\frac{v_{2n}^2}{2}+\cancel{\frac{v_{2t}^2}{2}}$$

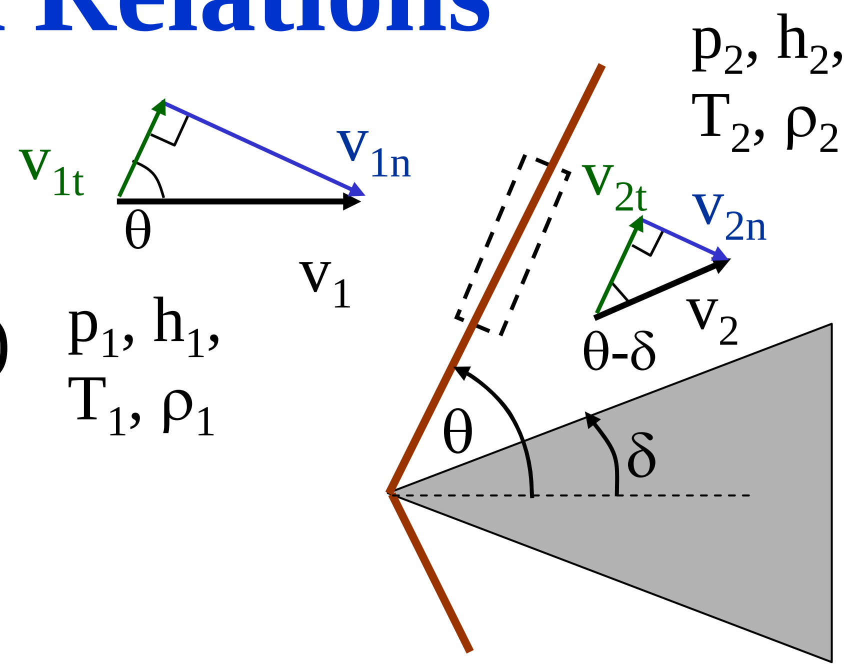
$$\boxed{h_1+\frac{v_{1n}^2}{2}=h_2+\frac{v_{2n}^2}{2}} \quad \textbf{(3)}$$



- Eq's. (1)-(3) are same equations used to characterize normal shocks (VII.1-3) with $v_n \to v$
- So oblique shock acts like normal shock in direction normal to wave
  - v_t constant, but $M_{t1} \neq M_{t2}$

$$\frac{M_{t2}}{M_{t1}}=\frac{\cancelto{1}{v_{t2}}/a_2}{\cancel{v_{t1}}/a_1} \Rightarrow \boxed{\frac{M_{t2}}{M_{t1}}=\sqrt{\frac{T_1}{T_2}}} \quad \textbf{(VII.20)}$$

Oblique Shock Relations



- To find conditions across shock, use M relations from normal shocks, e.g., (VII.5-17) but replace
  $M_1 \rightarrow M_1 \sin\theta = M_{1n}$
  $M_2 \rightarrow M_2 \sin(\theta\text{-}\delta) = M_{2n}$
- **Mach Number** (after shock)

from (VII.11) $$M_2^2 = \left(M_1^2 + \frac{2}{\gamma - 1}\right) \Big/ \left(\frac{2\gamma}{\gamma - 1} M_1^2 - 1\right)$$

$$M_2^2 \sin^2(\theta - \delta) = \left(M_1^2 \sin^2\theta + \frac{2}{\gamma - 1}\right) \Big/ \left(\frac{2\gamma}{\gamma - 1} M_1^2 \sin^2\theta - 1\right) \quad \textbf{(VII.21)}$$

Oblique Shock Relations (con't)

- **Static Properties**

(from VII.12)

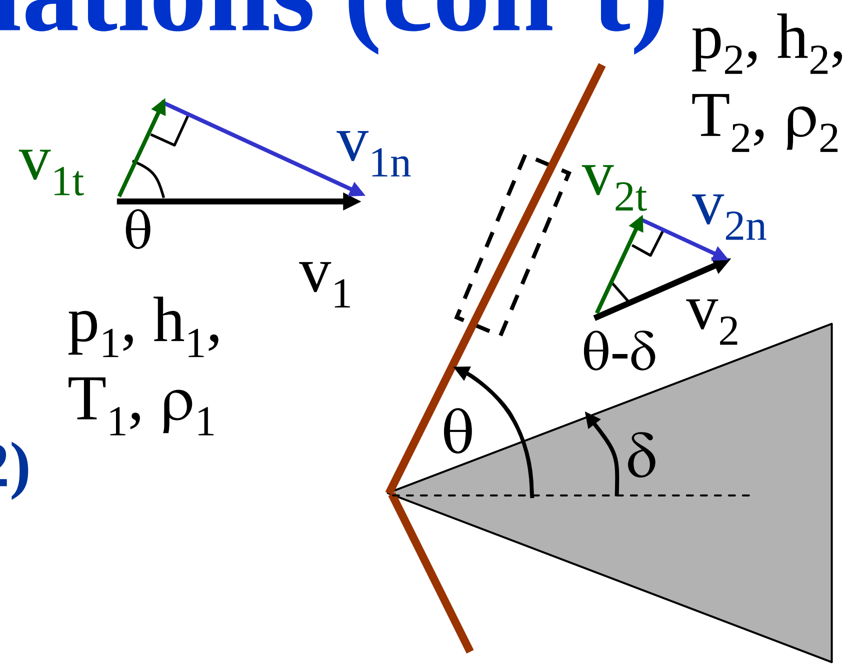
$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1} \qquad \textbf{(VII.22)}$$

(from VII.18)

$$\frac{v_{1n}}{v_{2n}} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2 \sin^2\theta}{(\gamma-1) M_1^2 \sin^2\theta + 2} \qquad \textbf{(VII.23)}$$

(from VII.9)

$$\frac{T_2}{T_1} = \frac{\left(1 + \frac{\gamma-1}{2} M_1^2 \sin^2\theta\right)\left(\frac{2\gamma}{\gamma-1} M_1^2 \sin^2\theta - 1\right)}{M_1^2 \sin^2\theta\,(\gamma+1)^2 / 2(\gamma-1)} \qquad \textbf{(VII.24)}$$



Oblique Shock Relations (con't)

- **Stagnation Properties**

T_o (from energy conservation)

$$T_{o2} = T_{o1}$$

p_o (from VII.14) $p_{o2}/p_{o1} = \left(T_1/T_2\right)^{\gamma/\gamma-1}\left(p_2/p_1\right)$

since function of static property ratios, don't have to factor in p_{ot} v. p_{on}

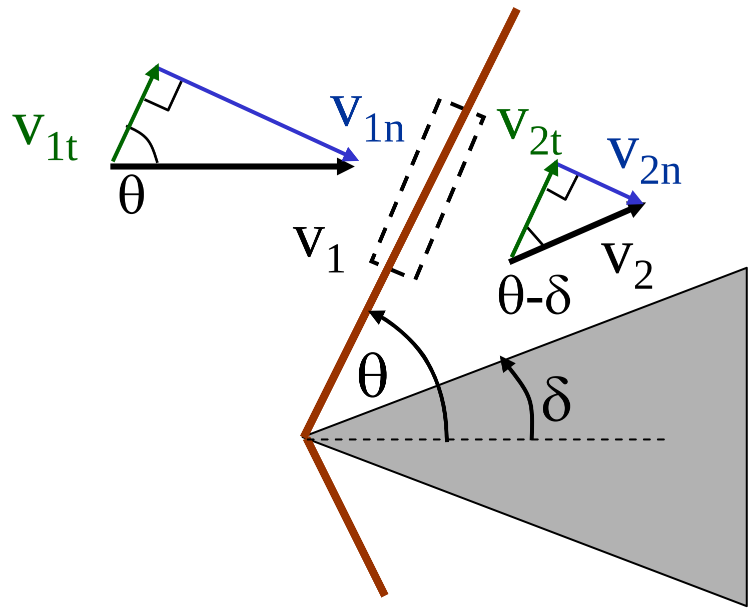
(from VII.13)

$$\frac{p_{o2}}{p_{o1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2\sin^2\theta}{1+\frac{\gamma-1}{2}M_1^2\sin^2\theta}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2\sin^2\theta-\frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}$$

(VII.25)

Wave/Shock Angle

- Equations above are functions of M_1, θ (shock angle) and δ (turning angle)
- Is there a relationship between them?



(from VII.23) (from geometry)

$$\frac{v_{1n}}{v_{2n}} = \frac{(\gamma+1)M_1^2\sin^2\theta}{(\gamma-1)M_1^2\sin^2\theta+2} = \frac{\cancel{v_{1t}}\tan\theta}{\cancel{v_{2t}}\tan(\delta-\theta)}$$

1

Alternate Eq'n.

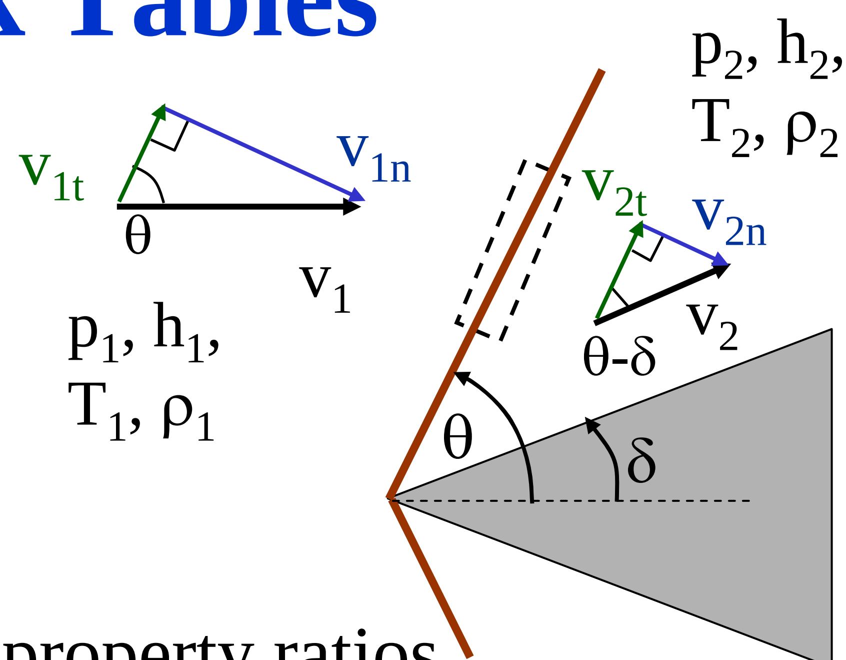
$$\tan\delta = \frac{(1/\tan\theta)(M_1^2\sin^2\theta-1)}{\frac{\gamma+1}{2}M_1^2-(M_1^2\sin^2\theta-1)}$$

$$\tan\delta = \frac{(2/\tan\theta)(M_1^2\sin^2\theta-1)}{M_1^2(\gamma+\cos 2\theta)+2}$$

(VII.26)

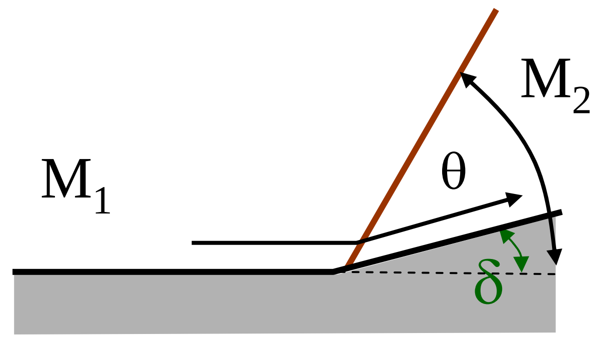
So to find oblique shock solution, need **2 indep. parameters**, e.g., M_1 and δ

Use of Shock Tables



- Since just replacing
  $M_1 \rightarrow M_1 \sin\theta$
  $M_2 \rightarrow M_2 \sin(\theta\text{-}\delta)$
  - can also use normal shock tables
  - use $M_1' = M_1 \sin\theta$ to look up property ratios
  - $M_2 = M_2'/\sin(\theta\text{-}\delta)$, with M_2' from normal shock tables
- **Warning**
  - do not use p_1/p_{o2} from tables
  - only gives p_{o2} associated with v_{2n}, not v_{2t}

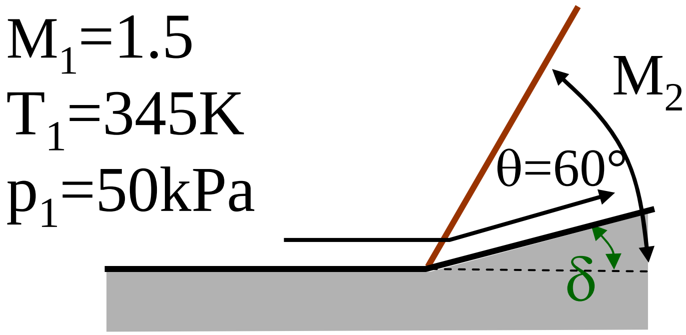
Oblique Solution Summary



- If given M_1 and shock angle, θ
  1. Find δ from VII.26 or use oblique shock charts (Appendix C in John)
  2. Calculate $M_{1n}=M_1\sin\theta$
  3. Use normal shock tables or Mach relations, e.g., VII.22-25 to get property ratios
  4. Get M_2 from $M_2=M_{2n}/\sin(\theta-\delta)$ or VII.21
- If given M_1 and turning angle, δ
  1. Find θ from (iteration) VII.26 or use oblique shock charts (e.g., Appendix C in John)
  2. Steps 2-4 above

Example #1 – Known Shock Angle

- **Given:** Uniform Mach 1.5 air flow (p=50 kPa, T=345K) approaching sharp concave corner. Oblique shock produced with shock angle of 60°



- **Find:**
  1. T_{o2}
  2. p_2
  3. δ (turning angle)

- **Assume:** TPG/CPG with γ=1.4, steady, adiabatic, no work, inviscid except for shock,….

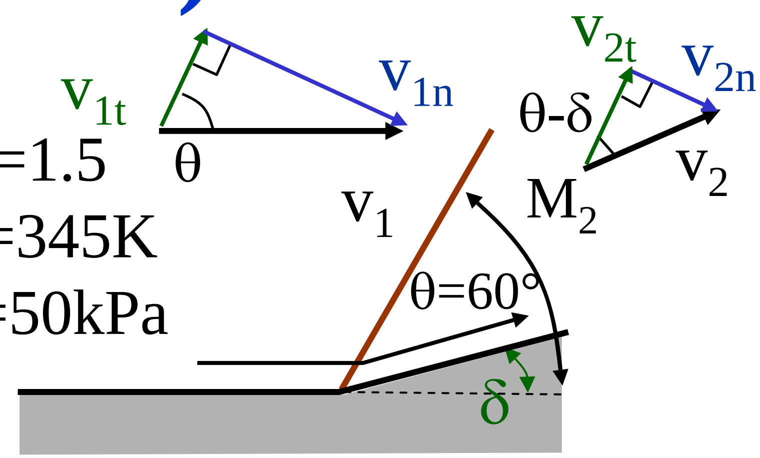
Example #1 (con't)

- **Analysis:**
  - T_o

$$T_{o2} = T_{o1} = T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right)$$

$$= 345\text{K}\left(1 + 0.2(1.5)^2\right) = \boxed{500\text{K}}$$

$M_1 = 1.5$
$T_1 = 345\text{K}$
$p_1 = 50\text{kPa}$



  - p_2
    - calculate normal component

$$M_{1n} = M_1 \sin\theta = 1.5 \sin 60^\circ = 1.30$$

$$(\text{B.1}) \Rightarrow M_{2n} = 0.786$$

$$p_2/p_1 = 1.805$$

$$T_2/T_1 = 1.191$$

$$p_2 = (p_2/p_1)p_1 = (1.805)50\text{kPa} = \boxed{90.3\text{kPa}}$$

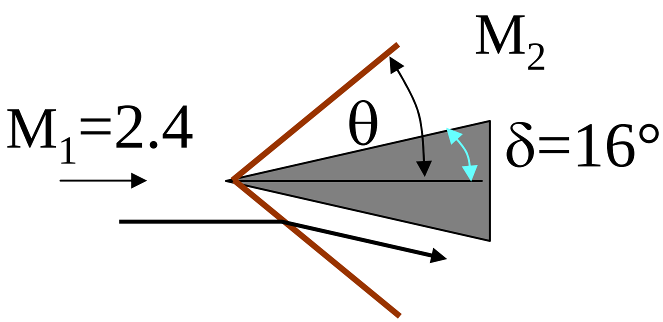
  - δ (from VII.26)

$$\tan\delta = \frac{\left(1/\tan 60^\circ\right)\left(1.5^2 \sin^2 60^\circ - 1\right)}{\frac{2.4}{2} 1.5^2 - \left(1.5^2 \sin^2 60^\circ - 1\right)} \Rightarrow \delta = \boxed{11.2^\circ}$$

NOTE: $M_2 = M_{2n}/\sin(\theta - \delta) = 1.04 > 1$ Supersonic flow okay after oblique shock

Example #2 – Known Turn Angle

- **Given:** Uniform Mach 2.4, cool, nitrogen flow passing over 2-d wedge with 16° half-angle.



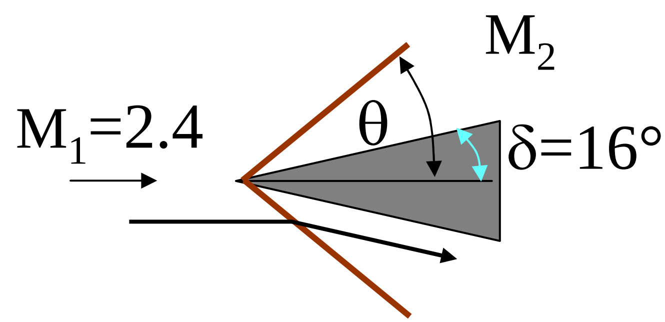
- **Find:**

θ, p_2/p_1 , T_2/T_1 , p_{o2}/p_{o1} , M_2

- **Assume:** N_2 is TPG/CPG with γ=1.4, steady, adiabatic, no work, inviscid except for shock,….

Example #2 (con't)

- **Analysis:**
  - θ (from VII.26)



$$\tan 16^\circ = \frac{(2/\tan\theta)\left(2.4^2 \sin^2\theta - 1\right)}{2.4^2(1.4 + \cos 2\theta) + 2}$$

iterate $\theta = 39.4^\circ$ or from John App. C: oblique shock charts

  - **use shock relations** calculate normal component

$$M_{1n} = M_1 \sin\theta = 2.4 \sin 39.4^\circ = 1.52$$

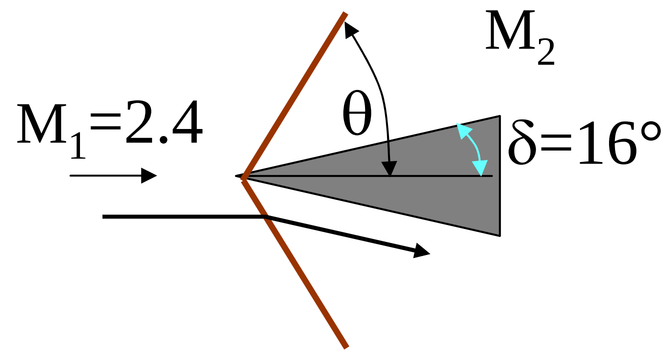
$$\text{(B.1)} \Rightarrow M_{2n} = 0.6935;\ p_2/p_1 = 2.535;\ T_2/T_1 = 1.335;\ p_{o2}/p_{o1} = 0.9227$$

$$M_2 = M_{2n}/\sin(\theta - \delta) = 1.75$$

Supersonic after shock

Example #2 (con't)

- **Analysis (con't):**
  - **can find second solution for θ**



$$\tan 16^\circ = \frac{(2/\tan\theta)\left(2.4^2\sin^2\theta - 1\right)}{2.4^2(1.4+\cos 2\theta)+2}$$

in addition to 39.4° $\theta = 82.1^\circ$

  - **use shock relations** calculate normal component

$$M_{1n} = M_1 \sin\theta = 2.4 \sin 82.1^\circ = 2.38$$

previous solution 2.535 1.335 0.9227

$$\text{(B.1)} \Rightarrow M_{2n} = 0.5256;\ p_2/p_1 = 6.425;\ T_2/T_1 = 2.018;\ p_{o2}/p_{o1} = 0.5499$$

1.75

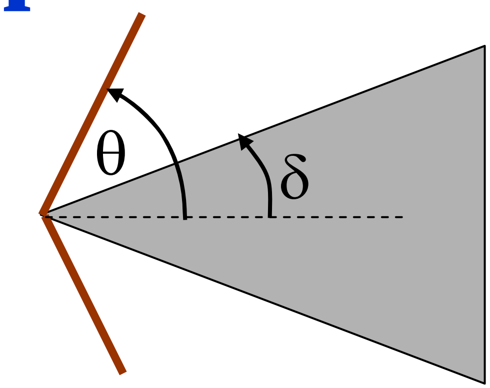
$$M_2 = M_{2n}/\sin(\theta-\delta) = 0.575$$

Now subsonic after shock

- VII.26 generally has **2 solutions for θ**: **Strong** and **Weak** oblique shocks

Alternate Approach

- Possible to find direct relationship* for θ as function of M_1 and δ



$$\tan\theta = \frac{M_1^2 - 1 + 2\lambda\cos\left(\dfrac{4\pi\alpha + \cos^{-1}\chi}{3}\right)}{3\left(1+\dfrac{\gamma-1}{2}M_1^2\right)\tan\delta} \quad \textbf{(VII.27)}$$

Removes iteration requirement…just longer equations

$$\lambda = \sqrt{\left(M_1^2-1\right)^2 - 3\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(1+\frac{\gamma+1}{2}M_1^2\right)\tan^2\delta}$$

$$\chi = \frac{1}{\lambda^3}\left[\left(M_1^2-1\right)^3 - 9\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(1+\frac{\gamma-1}{2}M_1^2+\frac{\gamma+1}{4}M_1^4\right)\tan^2\delta\right]$$

$$\alpha = \begin{cases} =0 & \text{strong shock solution} \\ =1 & \text{weak shock solution} \end{cases}$$

*e.g., Anderson, **Modern Comp. Flow**, 3rd ed., p. 142