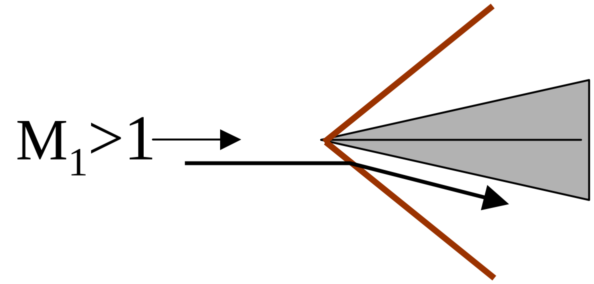
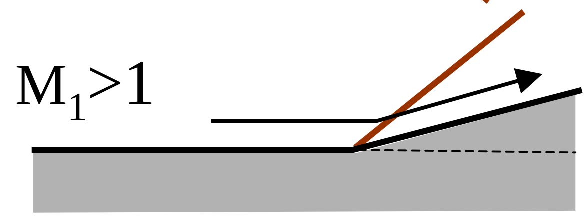
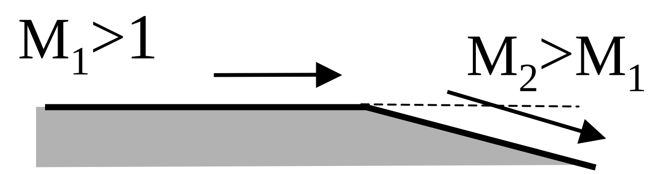
Supersonic Flow Turning

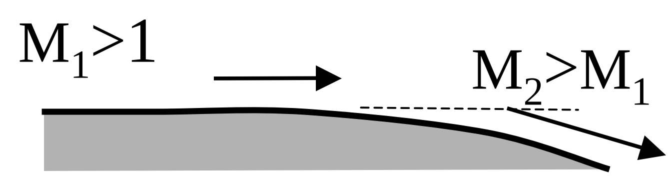
- Previously, we examined supersonic flow over (sharp) concave corners/turns
  - oblique shock allows flow to make this (compression) turn





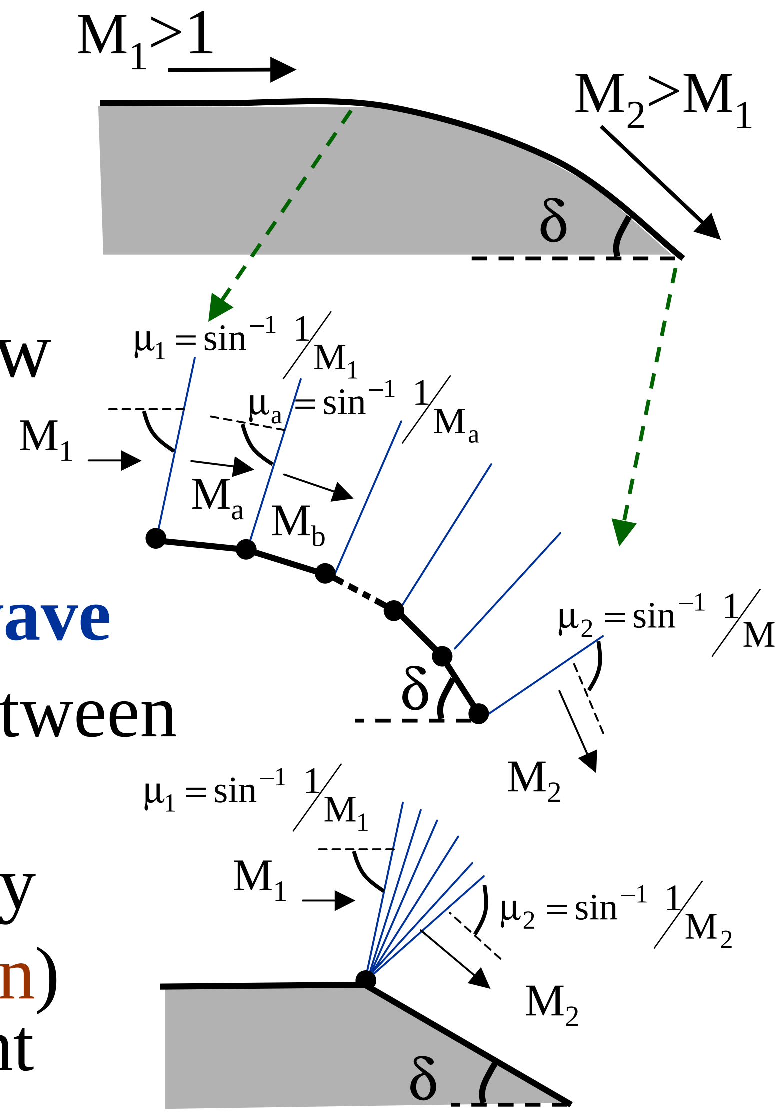
- What happens if:
  - turn is convex (expansion)
    - already shown expansion "shock" impossible (entropy would be destroyed)
  - turn is gradual (concave or convex)





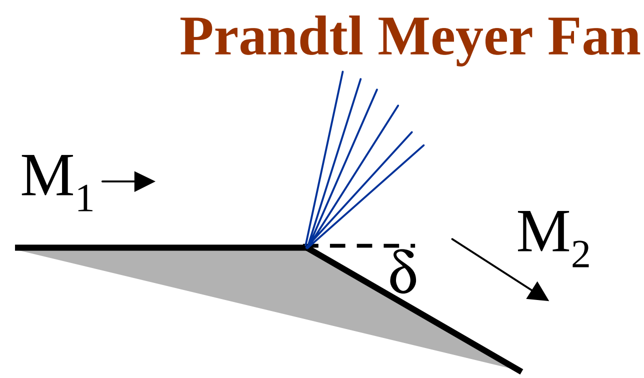
Gradual Expansion Turn

- Gradual turn is made up of large number of infinitessimal turns/corners
- Each turn has infinitessimal flow change
  - each turn produced by infinitessimal wave ⇒ **Mach wave**
- Flow is uniform and isentropic between each turn/corner
  - length between each is arbitrary
  - could be zero length (sharp turn) and waves collapse to one point



Prandtl Meyer Expansion Fan

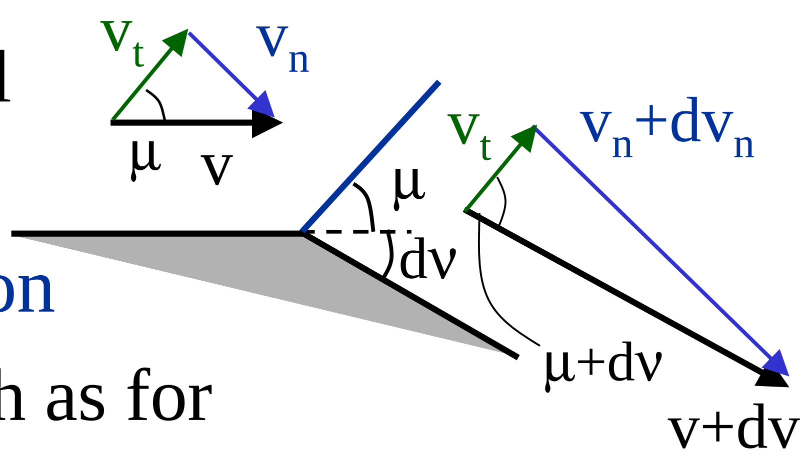
- Problem
  - given upstream conditions (1) and turning angle (δ)
  - find downstream conditions (2)



- Goal
  - Mach number relations (similar to shock relations)
- Equations
  - use mass, momentum, energy conservation, Mach number def'n., state equations
- Assumptions
  - steady flow, quasi-1d, reversible+adiabatic (isentropic)

Mach Relations

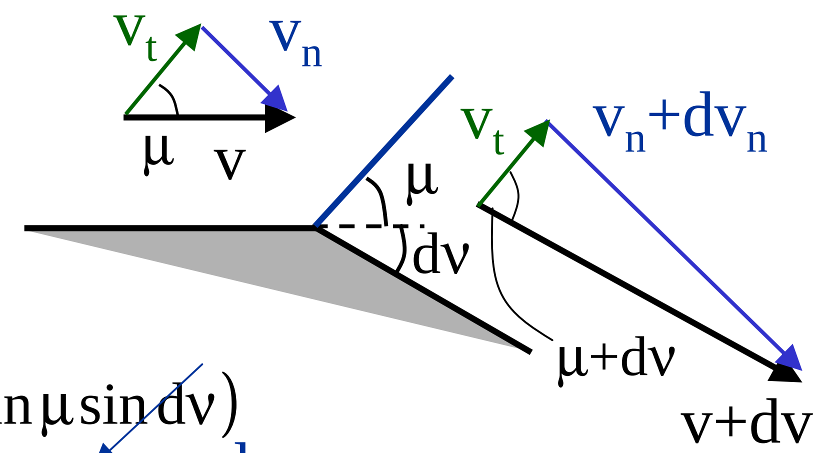
- Approach
  - begin with single Mach wave that expands supersonic flow through an infinitessimal (differential) angle of magnitude $d\nu$
  - essentially using differential control volume



- Mass/Momentum Conservation
  - using same type of approach as for oblique shocks (two momentum components: t, n)
  - find lack of pressure gradient tangent to wave gives **v_t=constant across wave**

Relation Between Velocity and Angles

- Use v_t=constant



$$v_{t,upstream} = v_{t,downstream}$$

$$v\cos\mu = (v+dv)\cos(\mu+d\nu)$$

$$= (v+dv)(\cos\mu\cos d\nu - \sin\mu\sin d\nu)$$

$d\nu \to 0$: $\cos d\nu \to 1$, $\sin d\nu \to d\nu$

$$\cancel{v\cos\mu} = \cancel{v\cos\mu} - v\,d\nu\sin\mu + dv\cos\mu - dv\,d\nu\sin\mu$$

($dv\,d\nu \to 0$)

$$\frac{dv}{v} = \frac{\sin\mu}{\cos\mu}d\nu$$

$\sin\mu = 1/M$
$\sin^2\mu + \cos^2\mu = 1$

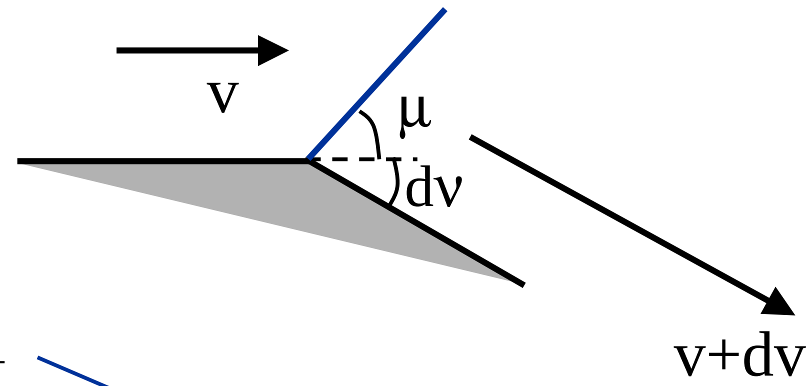
$$\frac{dv}{v} = \sqrt{\frac{1/M^2}{1-1/M^2}}\,d\nu \;\longrightarrow\; \frac{dv}{v} = \frac{1}{\sqrt{M^2-1}}\,d\nu \qquad \textbf{(VIII.1)}$$

Relation Between M and dν

- **Relate v and M**

v=Ma

$$\frac{dv}{v}=\frac{dM}{M}+\frac{da}{a}$$



tpg/cpg
$a^2=\gamma RT$

$$\frac{dv}{v}=\frac{dM}{M}+\frac{d\sqrt{T}}{\sqrt{T}}=\frac{dM}{M}+\frac{1}{2}\frac{dT}{T}$$

Energy Conservation

$$T_o=T\left(1+\frac{\gamma-1}{2}M^2\right)=\text{const.}$$

$$\frac{dT_o}{T_o}=0=\frac{dT}{T}+\frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

$$\frac{dT}{T}=-\frac{(\gamma-1)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

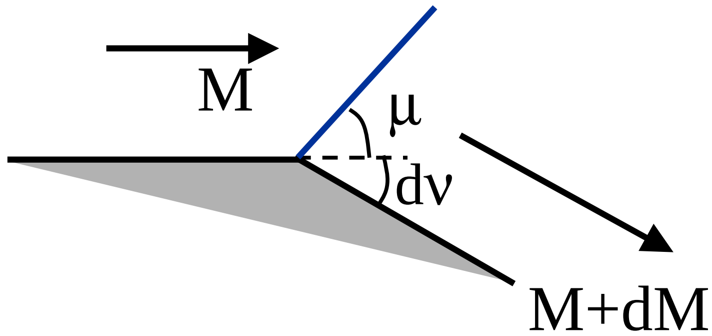
$$\frac{dv}{v}=\frac{dM}{M}-\frac{[(\gamma-1)/2]M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

$$\frac{dv}{v}=\frac{dM}{M}\left[1-\frac{\frac{(\gamma-1)}{2}M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]=\frac{dM}{M}\left[\frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]$$

Relation Between M and dν (con't)

- **Relate VIII.1 and last eqn.**

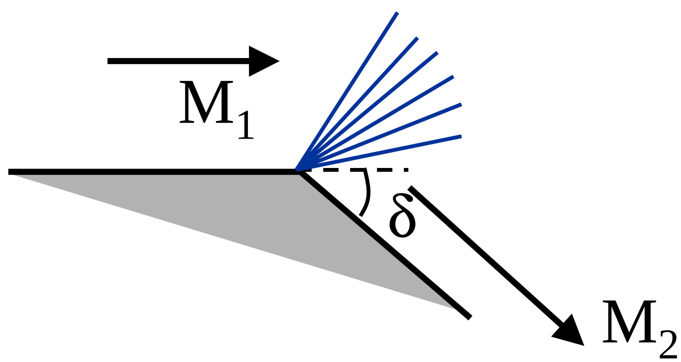
$$\frac{dv}{v} = \frac{1}{\sqrt{M^2-1}}d\nu = \frac{dM}{M}\left[\frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]$$



$$d\nu = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$ **(VIII.2)**

dM is change in Mach number associated with dν turn angle

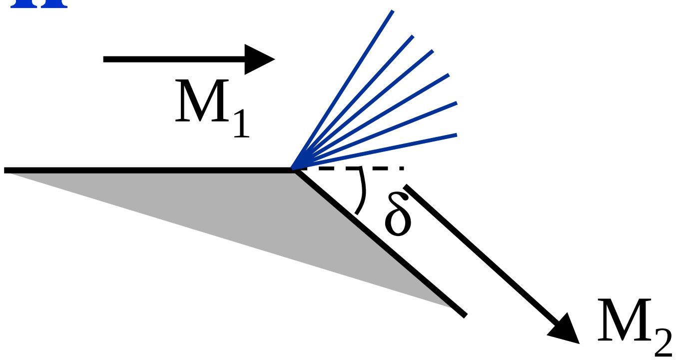
- Need finite angle, $\delta=\nu_2-\nu_1$ and finite ΔM



integrate →

$$\nu_2-\nu_1 = \int_{\nu_1}^{\nu_2} d\nu = \int_{M_1}^{M_2}\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

M - ν Relation



- Perform Integration

$$\nu_2 - \nu_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$\nu_2 - \nu_1 = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}\right]_{M_1}^{M_2} \quad \textbf{(VIII.3)}$$

- So, given δ (=ν_2-ν_1) and M_1
  - could "solve" VIII.3 for M_2
- Can not invert VIII.3 analytically (M_2=f(M_1, δ))
  - either use *interative* (e.g., numerical or guessing) method
  - or *find ν as a function of M* and tabularize or graph solution

Tabular Solutions/Reference Condition

$$\nu_2 - \nu_1 = \left[\sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1} \right]_{M_1}^{M_2}$$

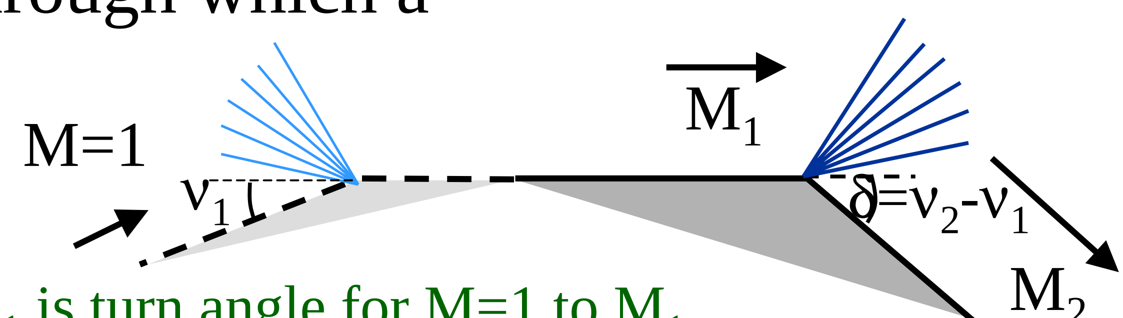
- Want to find $\nu=\nu(M)$ [really $\nu_2=\nu_2(M_2)$] for any M
  - need to choose (arbitrary) reference condition, i.e., pick an M where $\nu=0$
  - let's choose $\nu=0$ at M=1

- analagous to table of h(T)
  - really $h(T)-h(T_{ref})$
  - just chosen $h(T_{ref})=0$

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1} \qquad \textbf{(VIII.4)}$$

Appendix D (John) for $\gamma=1.4$, $M\leq5$

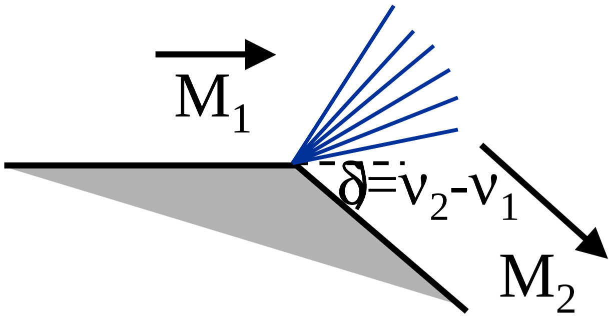
- ν represents angle through which a sonic flow would have to turn to reach M



ν_1 is turn angle for M=1 to M_1

Using the Prandtl Meyer Tables

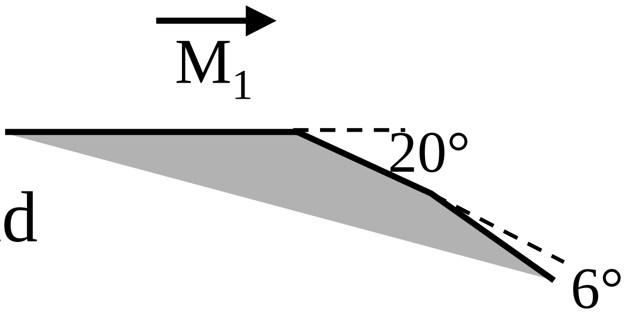
- To find M_2 given M_1 and δ
  - find ν_1 (for given M_1) from table
  - get ν_2 from $\delta=\nu_2-\nu_1$
  - look up ν_2 in table to find M_2



- To find T_2, p_2, ...
  - use isentropic flow relations since expansion is isentropic (no shock)
  - e.g.,

$$\frac{T_o}{T}=1+\frac{\gamma-1}{2}M^2 \;\xrightarrow{T_o=\text{const}}\; \frac{T_2}{T_1}=\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}$$

Example



- **Given:** Uniform Mach 2 flow of nitrogen at 300 K flows over compound wall corner: two turns, 20° and 6°

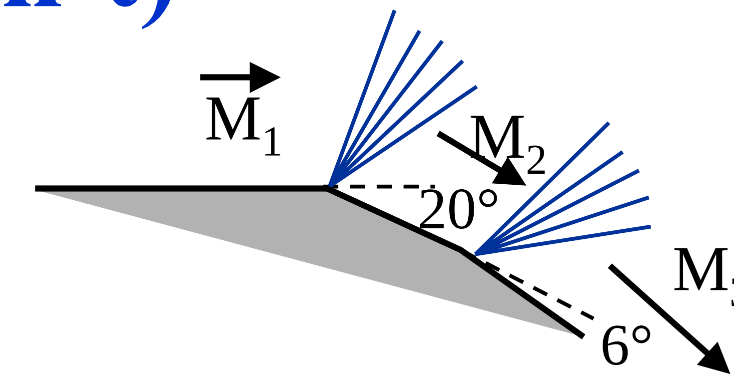
- **Find:**

  M and T after final turn

- **Assume:** N_2 is TPG/CPG with γ=1.4, steady, adiabatic, no work, inviscid,….

Example (con't)

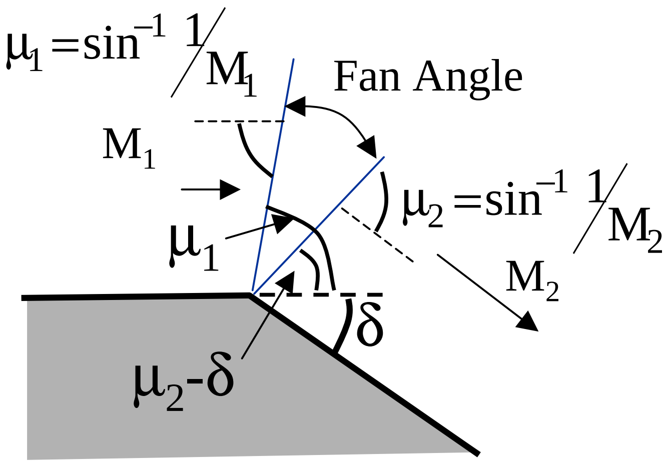
- **Analysis: (class exercise)**
  - To find M_2 given M_1 and δ
    1. find ν_1 (for given M_1) from table
    2. get ν_2 from $\delta=\nu_2-\nu_1$
    3. look up ν_2 in table to find M_2



M	ν	M	ν	M	ν	M	ν	M	ν
2.00	26.38	2.60	41.41	2.75	44.69	2.90	47.79	3.05	50.71
2.01	26.66	2.61	41.64	2.76	44.91	2.91	47.99	3.06	50.90
2.02	26.93	2.62	41.86	2.77	45.12	2.92	48.19	3.07	51.09
2.03	27.20	2.63	42.09	2.78	45.33	2.93	48.39	3.08	51.28
2.04	27.48	2.64	42.31	2.79	45.54	2.94	48.59	3.09	51.46
2.50	39.12	2.65	42.53	2.80	45.75	2.95	48.78	3.10	51.65
2.51	39.36	2.66	42.75	2.81	45.95	2.96	48.98	3.11	51.83
2.52	39.59	2.67	42.97	2.82	46.16	2.97	49.18	3.12	52.02
2.53	39.82	2.68	43.19	2.83	46.37	2.98	49.37	3.13	52.20
2.54	40.05	2.69	43.40	2.84	46.57	2.99	49.56	3.14	52.39
2.55	40.28	2.70	43.62	2.85	46.78	3.00	49.76	3.15	52.57
2.56	40.51	2.71	43.84	2.86	46.98	3.01	49.95	3.16	52.75
2.57	40.74	2.72	44.05	2.87	47.19	3.02	50.14	3.17	52.93
2.58	40.96	2.73	44.27	2.88	47.39	3.03	50.33	3.18	53.11
2.59	41.19	2.74	44.48	2.89	47.59	3.04	50.52	3.19	53.29

Prandtl Meyer Fan Angle

- **Fan angle**
  - angle between first and last Mach wave
  - useful to determine when expansion has ended in flowfield for a given distance away from wall



- From geometry

$$\text{Fan Angle} = \mu_1 - (\mu_2 - \delta)$$

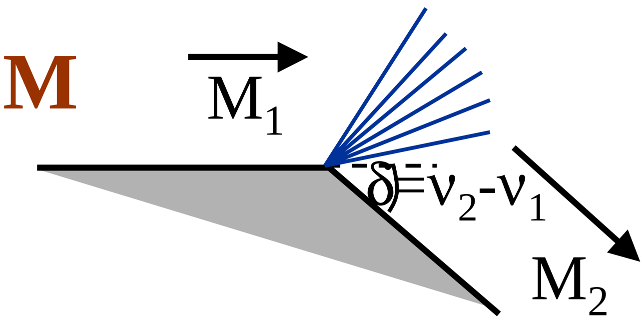
$$= (\mu_1 - \mu_2) + \delta$$

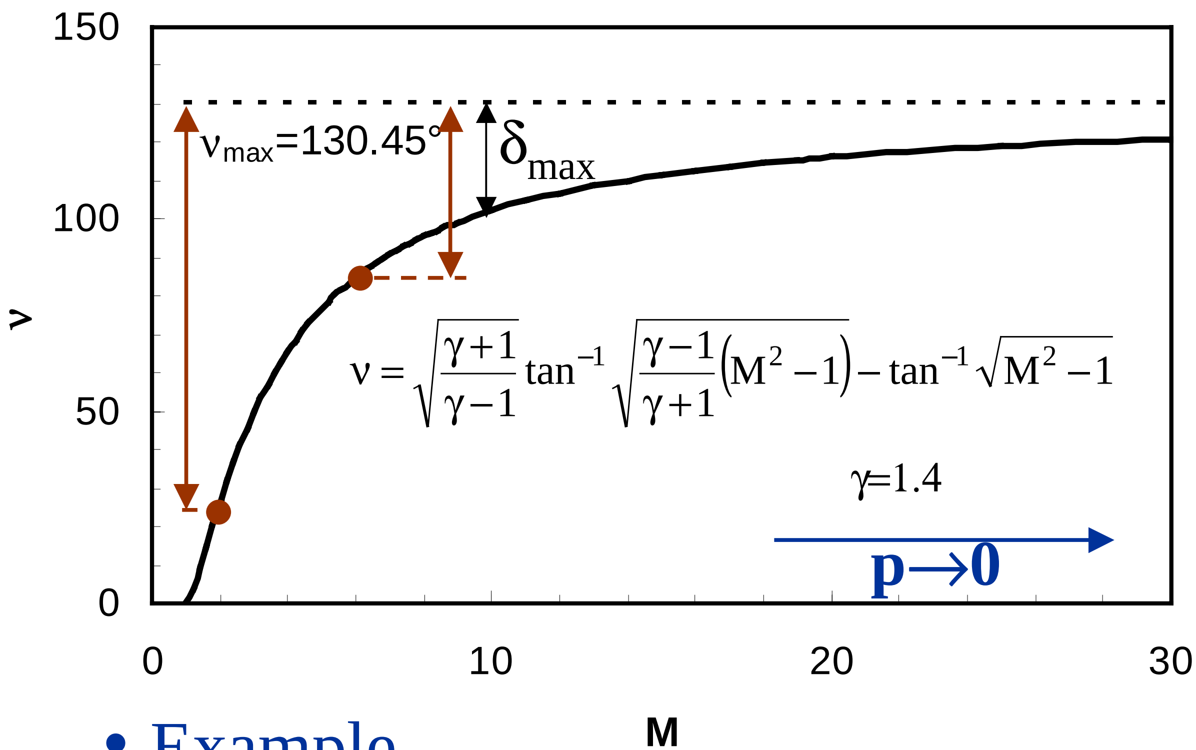
(VIII.5)

$$= (\mu_1 - \mu_2) + (\nu_2 - \nu_1)$$

Prandtl Meyer Turns at High M

- **Examine plot of ν as function of M**



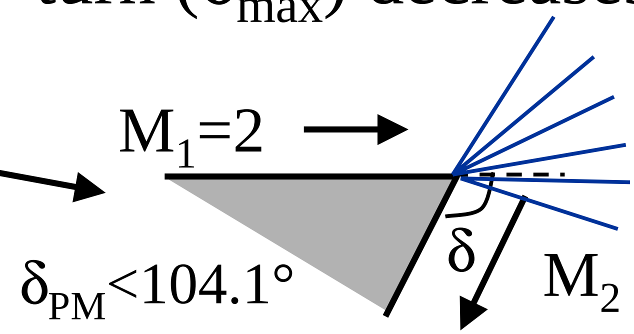


- As M increases, reach **maximum turn angle** (ν_{max}~130.5° for γ=1.4)
- So as M_1 increases, max. angle flow can turn (δ_{max}) decreases

- Example

$$M_1 = 2 \Rightarrow \delta_{max} = 130.45^\circ - 26.38^\circ = 104.1^\circ$$

$$M_1 = 6 \Rightarrow \delta_{max} = 130.45^\circ - 84.96^\circ = 45.5^\circ$$



Maximum Turn Angle

- **Analytic Expression**

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}$$

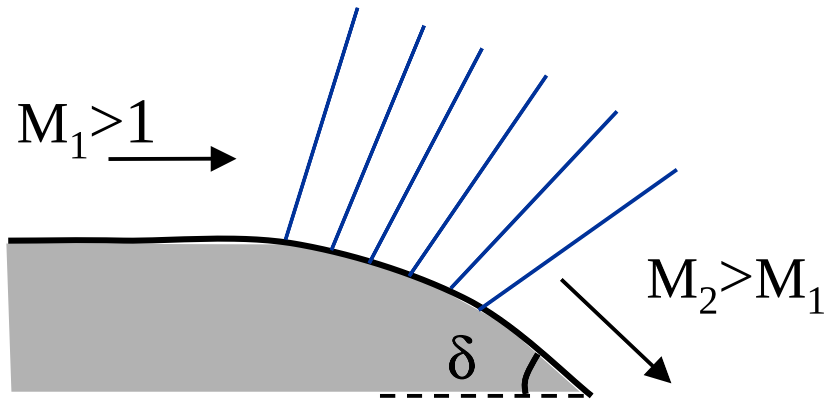
$$\text{For } M \to \infty : \sqrt{\ } \to M; \tan^{-1}(M) \to 90^\circ$$

$$\nu_{max} = \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right) 90^\circ \qquad \textbf{(VIII.6)}$$

$$\nu_{max} = \begin{cases} 90^\circ & \gamma = 5/3 \\ 130.45^\circ & \gamma = 1.4 \\ 159.2^\circ & \gamma = 1.3 \\ 208.5^\circ & \gamma = 1.2 \end{cases}$$

- As γ decreases (higher temperatures, bigger molecules), **maximum turn angle** increases
- δ_{max} smaller in real flows
  - T and p drop through turn
    - ⇒condensation of gas
    - ⇒nonequilibrium flow

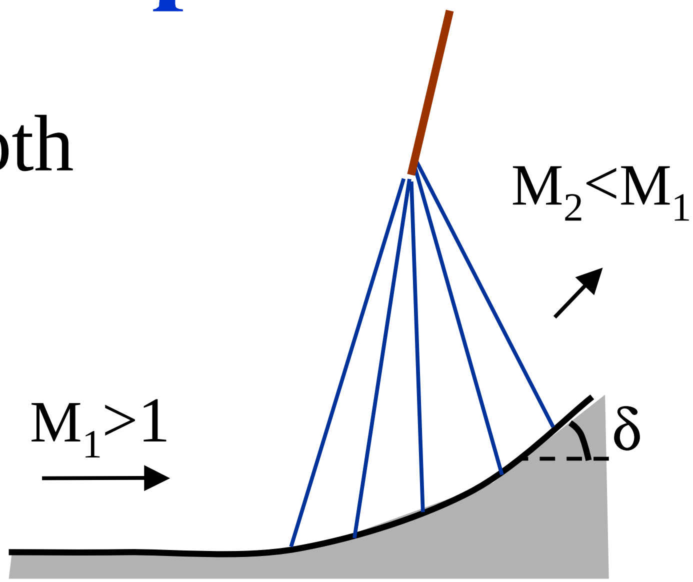
Continuous (Smooth) Expansions



- Already showed that it does not matter if expansion turn is sharp or smooth
  - still get same solution, P-M fan=infinite set of Mach waves
  - unless we exceed the maximum turning angle, final properties just function of total turn angle
  - smooth turn just means expansion process takes place over longer distance

Continuous (Smooth) Compressions

- What happens if we have a smooth concave turn?
  - since flow direction change is small, can still get set of weak Mach waves



Prandtl-Meyer compression: **$\delta = -(\nu_2 - \nu_1)$, so $\nu_2 - \nu_1 < 0$**

  - however unlike expansions, compressions merge
  - together they coalesce to form **oblique shock**
  - flow that went through PM compression is isentropic, outer flow has entropy rise (p_o loss)
  - size of PM region depends on M_1 and curvature