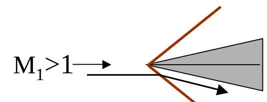
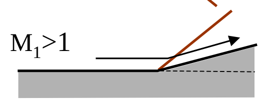
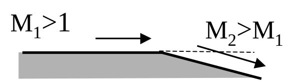
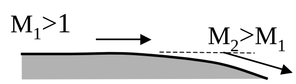
# Supersonic Flow Turning

- Previously, we examined supersonic flow over (sharp) concave corners/turns
  - oblique shock allows flow to make this (compression) turn
- What happens if:
  - turn is convex (expansion)
    - already shown expansion "shock" impossible (entropy would be destroyed)
  - turn is gradual (concave or convex)



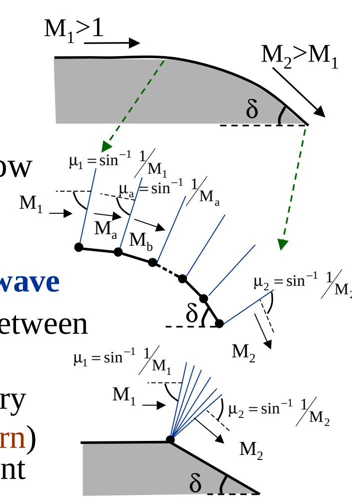






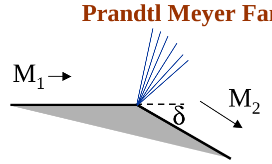

# Gradual Expansion Turn

- Gradual turn is made up of large number of infinitessimal turns/corners
- Each turn has infinitessimal flow change
  - each turn produced by infinitessimal wave ⇒ **Mach wave**
- Flow is uniform and isentropic between each turn/corner
  - length between each is arbitrary
  - could be zero length (sharp turn) and waves collapse to one point



## Prandtl Meyer Expansion Fan

- Problem
  - given upstream conditions (1) and turning angle (δ)
  - find downstream conditions (2)



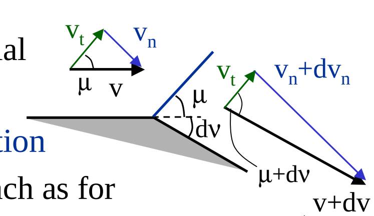

- Goal
  - Mach number relations (similar to shock relations)
- Equations
  - use mass, momentum, energy conservation, Mach number def'n., state equations
- Assumptions
  - steady flow, quasi-1d, reversible+adiabatic (isentropic)





## Mach Relations

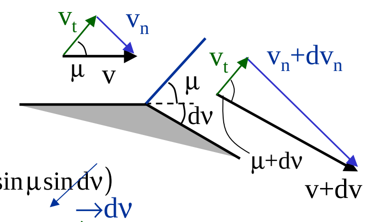
- Approach
  - begin with single Mach wave that expands supersonic flow through an infinitessimal (differential) angle of magnitude dν
  - essentially using differential control volume




- Mass/Momentum Conservation
  - using same type of approach as for oblique shocks (two momentum components: t, n)
  - find lack of pressure gradient tangent to wave gives **$v_t$=constant across wave**

# Relation Between Velocity and Angles

- **Use $v_t$=constant**




$$v_{t,upstream} = v_{t,downstream}$$

$$v\cos\mu = (v+dv)\cos(\mu+d\nu)$$

$$= (v+dv)(\cos\mu \underbrace{\cos d\nu}_{\to 1} - \sin\mu \underbrace{\sin d\nu}_{\to d\nu})$$

$d\nu \to 0$

$$\cancel{v\cos\mu} = \cancel{v\cos\mu} - v\,d\nu\sin\mu + dv\cos\mu - \underbrace{dv\,d\nu}_{\to 0}\sin\mu$$

$$\frac{dv}{v} = \frac{\sin\mu}{\cos\mu}d\nu$$

$\sin\mu=1/M$
$\sin^2\mu+\cos^2\mu=1$

$$\frac{dv}{v} = \sqrt{\frac{1/M^2}{1-1/M^2}}\,d\nu \longrightarrow \boxed{\frac{dv}{v} = \frac{1}{\sqrt{M^2-1}}d\nu} \qquad \textbf{(VIII.1)}$$

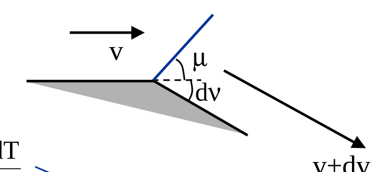






# Relation Between M and dν

- **Relate v and M**




$v=Ma$
$$\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}$$

tpg/cpg
$a^2=\gamma RT$
$$\frac{dv}{v} = \frac{dM}{M} + \frac{d\sqrt{T}}{\sqrt{T}} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

**Energy Conservation**

$$T_o = T\left(1+\frac{\gamma-1}{2}M^2\right) = \text{const.}$$

$$\frac{dT_o}{T_o} = 0 = \frac{dT}{T} + \frac{d\left(1+\frac{\gamma-1}{2}M^2\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}$$

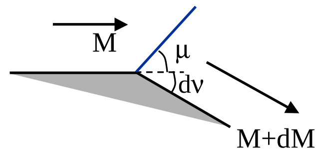
$$\frac{dT}{T} = -\frac{(\gamma-1)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

$$\frac{dv}{v} = \frac{dM}{M} - \frac{[(\gamma-1)/2]M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

$$\frac{dv}{v} = \frac{dM}{M}\left[1-\frac{\frac{(\gamma-1)}{2}M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right] = \frac{dM}{M}\left[\frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]$$

# Relation Between M and dν (con't)

- **Relate VIII.1 and last eqn.**

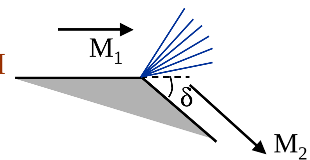



$$\frac{d\nu}{\nu} = \frac{1}{\sqrt{M^2-1}}\, d\nu = \frac{dM}{M}\left[\frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]$$

$$d\nu = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \qquad \textbf{(VIII.2)}$$

dM is change in Mach number associated with dν turn angle

- Need finite angle, $\delta=\nu_2-\nu_1$ and finite ΔM



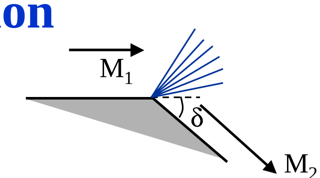

**integrate** $\nu_2-\nu_1 = \int_{\nu_1}^{\nu_2} d\nu = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$





# M - ν Relation

- Perform Integration




$$\nu_2-\nu_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

$$\nu_2-\nu_1 = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1}\right]_{M_1}^{M_2} \qquad \textbf{(VIII.3)}$$

- So, given δ ($=\nu_2-\nu_1$) and $M_1$
  - could "solve" VIII.3 for $M_2$
- Can not invert VIII.3 analytically ($M_2=f(M_1, \delta)$)
  - either use *interative* (e.g., numerical or guessing) method
  - or *find ν as a function of M* and tabularize or graph solution

# Tabular Solutions/Reference Condition

$$\nu_2-\nu_1=\left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1}\right]_{M_1}^{M_2}$$

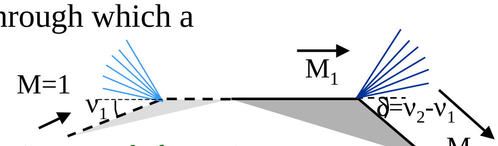
- Want to find $\nu=\nu(M)$ [really $\nu_2=\nu_2(M_2)$] for any M
  - need to choose (arbitrary) reference condition, i.e., pick an M where $\nu=0$
  - let's choose $\nu=0$ at M=1

• analagous to table of h(T)
– really $h(T)-h(T_{ref})$
– just chosen $h(T_{ref})=0$

$$\nu=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1}$$

**(VIII.4)** Appendix D (John) for $\gamma$=1.4, M≤5

- $\nu$ represents angle through which a sonic flow would have to turn to reach M




$\nu_1$ is turn angle for M=1 to $M_1$





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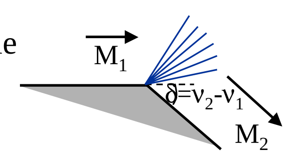
# Using the Prandtl Meyer Tables

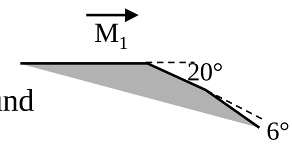
- To find $M_2$ given $M_1$ and $\delta$
  - find $\nu_1$ (for given $M_1$) from table
  - get $\nu_2$ from $\delta=\nu_2-\nu_1$
  - look up $\nu_2$ in table to find $M_2$




- To find $T_2$, $p_2$, ....
  - use isentropic flow relations since expansion is isentropic (no shock)
  - e.g.,

$$\frac{T_o}{T}=1+\frac{\gamma-1}{2}M^2 \;\xRightarrow{T_o=\text{const}}\; \frac{T_2}{T_1}=\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}$$

# Example




- **Given:** Uniform Mach 2 flow of nitrogen at 300 K flows over compound wall corner: two turns, 20° and 6°
- **Find:**

  M and T after final turn

- **Assume:** $N_2$ is TPG/CPG with $\gamma$=1.4, steady, adiabatic, no work, inviscid,….

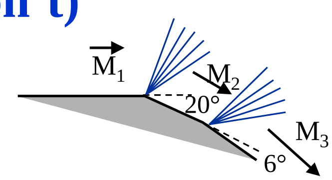






# Example (con't)

- **Analysis: (class exercise)**
  - To find $M_2$ given $M_1$ and $\delta$
    1. find $\nu_1$ (for given $M_1$) from table
    2. get $\nu_2$ from $\delta=\nu_2-\nu_1$
    3. look up $\nu_2$ in table to find $M_2$




| M | ν | M | ν | M | ν | M | ν | M | ν |
|---|---|---|---|---|---|---|---|---|---|
| 2.00 | 26.38 | 2.60 | 41.41 | 2.75 | 44.69 | 2.90 | 47.79 | 3.05 | 50.71 |
| 2.01 | 26.66 | 2.61 | 41.64 | 2.76 | 44.91 | 2.91 | 47.99 | 3.06 | 50.90 |
| 2.02 | 26.93 | 2.62 | 41.86 | 2.77 | 45.12 | 2.92 | 48.19 | 3.07 | 51.09 |
| 2.03 | 27.20 | 2.63 | 42.09 | 2.78 | 45.33 | 2.93 | 48.39 | 3.08 | 51.28 |
| 2.04 | 27.48 | 2.64 | 42.31 | 2.79 | 45.54 | 2.94 | 48.59 | 3.09 | 51.46 |
| 2.50 | 39.12 | 2.65 | 42.53 | 2.80 | 45.75 | 2.95 | 48.78 | 3.10 | 51.65 |
| 2.51 | 39.36 | 2.66 | 42.75 | 2.81 | 45.95 | 2.96 | 48.98 | 3.11 | 51.83 |
| 2.52 | 39.59 | 2.67 | 42.97 | 2.82 | 46.16 | 2.97 | 49.18 | 3.12 | 52.02 |
| 2.53 | 39.82 | 2.68 | 43.19 | 2.83 | 46.37 | 2.98 | 49.37 | 3.13 | 52.20 |
| 2.54 | 40.05 | 2.69 | 43.40 | 2.84 | 46.57 | 2.99 | 49.56 | 3.14 | 52.39 |
| 2.55 | 40.28 | 2.70 | 43.62 | 2.85 | 46.78 | 3.00 | 49.76 | 3.15 | 52.57 |
| 2.56 | 40.51 | 2.71 | 43.84 | 2.86 | 46.98 | 3.01 | 49.95 | 3.16 | 52.75 |
| 2.57 | 40.74 | 2.72 | 44.05 | 2.87 | 47.19 | 3.02 | 50.14 | 3.17 | 52.93 |
| 2.58 | 40.96 | 2.73 | 44.27 | 2.88 | 47.39 | 3.03 | 50.33 | 3.18 | 53.11 |
| 2.59 | 41.19 | 2.74 | 44.48 | 2.89 | 47.59 | 3.04 | 50.52 | 3.19 | 53.29 |

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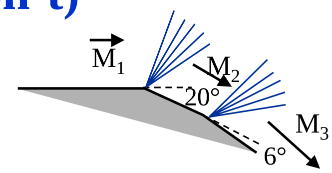
# Example (con't)

- **Analysis: (solution)**





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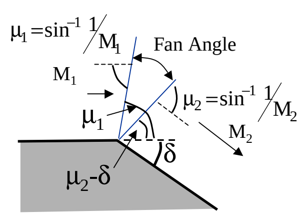
# Prandtl Meyer Fan Angle

- **Fan angle**
  - angle between first and last Mach wave
  - useful to determine when expansion has ended in flowfield for a given distance away from wall
- From geometry




$$\text{Fan Angle} = \mu_1 - (\mu_2 - \delta)$$

$$= (\mu_1 - \mu_2) + \delta \qquad \textbf{(VIII.5)}$$

$$= (\mu_1 - \mu_2) + (\nu_2 - \nu_1)$$

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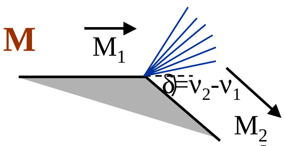
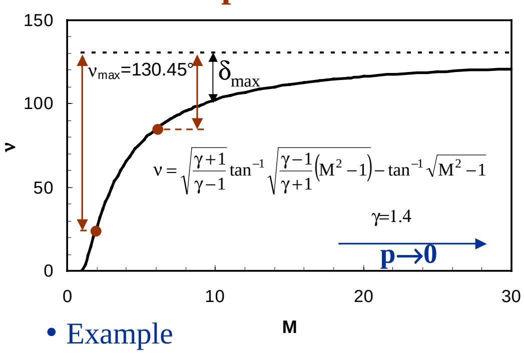
# Prandtl Meyer Turns at High M

- **Examine plot of ν as function of M**



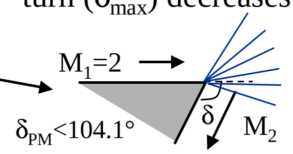




$$\nu=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1}$$

- As M increases, reach **maximum turn angle** ($\nu_{max}$~130.5° for $\gamma$=1.4)
- So as $M_1$ increases, max. angle flow can turn ($\delta_{max}$) decreases

- Example

$$M_1=2\Rightarrow\delta_{max}=130.45^\circ-26.38^\circ=104.1^\circ$$
$$M_1=6\Rightarrow\delta_{max}=130.45^\circ-84.96^\circ=45.5^\circ$$




Prandtl Meyer -15



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# Maximum Turn Angle

- **Analytic Expression**

$$\nu=\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}-\tan^{-1}\sqrt{M^2-1}$$

For $M\to\infty:\sqrt{\ }\to M;\tan^{-1}(M)\to 90^\circ$

$$\nu_{max}=\left(\sqrt{\frac{\gamma+1}{\gamma-1}}-1\right)90^\circ \qquad \textbf{(VIII.6)}$$

$$\nu_{max}=\begin{cases}90^\circ & \gamma=5/3\\ 130.45^\circ & \gamma=1.4\\ 159.2^\circ & \gamma=1.3\\ 208.5^\circ & \gamma=1.2\end{cases}$$

- As γ decreases (higher temperatures, bigger molecules), **maximum turn angle** increases
- $\delta_{max}$ smaller in real flows
  - T and p drop through turn
    - ⇒condensation of gas
    - ⇒nonequilibrium flow

Prandtl Meyer -16



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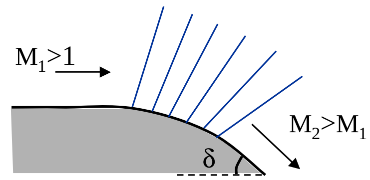
# Continuous (Smooth) Expansions




- Already showed that it does not matter if expansion turn is sharp or smooth
  - still get same solution, P-M fan=infinite set of Mach waves
  - unless we exceed the maximum turning angle, final properties just function of total turn angle
  - smooth turn just means expansion process takes place over longer distance





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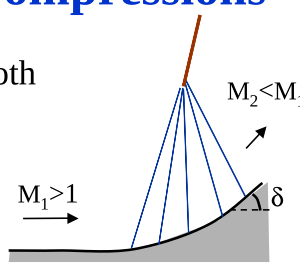
# Continuous (Smooth) Compressions




- What happens if we have a smooth concave turn?
  - since flow direction change is small, can still get set of weak Mach waves

**Prandtl-Meyer compression:** $\delta=-(\nu_2-\nu_1)$, so $\nu_2-\nu_1<0$

  - however unlike expansions, compressions merge
  - together they coalesce to form **oblique shock**
  - flow that went through PM compression is isentropic, outer flow has entropy rise ($p_o$ loss)
  - size of PM region depends on $M_1$ and curvature