Supersonic Flow Turning

- Previously, we examined supersonic flow over (sharp) concave corners/turns
  - oblique shock allows flow to make this (compression) turn

- What happens if:
  - turn is convex (expansion)
    - already shown expansion “shock” impossible (entropy would be destroyed)
  - turn is gradual (concave or convex)

Gradual Expansion Turn

- Gradual turn is made up of large number of infinitesimal turns/corners
- Each turn has infinitesimal flow change
  - each turn produced by infinitesimal wave \( \Rightarrow \text{Mach wave} \)
- Flow is uniform and isentropic between each turn/corner
  - length between each is arbitrary
  - could be zero length (sharp turn) and waves collapse to one point
Prandtl Meyer Expansion Fan

• Problem
  – given upstream conditions (1)
    and turning angle (δ)
  – find downstream conditions (2)

• Goal
  – Mach number relations (similar to shock relations)

• Equations
  – use mass, momentum, energy conservation, Mach
    number def’n., state equations

• Assumptions
  – steady flow, quasi-1d, reversible+adiabatic (isentropic)

Mach Relations

• Approach
  – begin with single Mach wave that expands supersonic
    flow through an infinitesimal (differential) angle of
    magnitude dv
  – essentially using differential
    control volume

• Mass/Momentum Conservation
  – using same type of approach as for
    oblique shocks (two momentum components: t, n)
  – find lack of pressure gradient tangent to wave gives
    \( v_t = \text{constant across wave} \)
Relation Between Velocity and Angles

- Use $v_t =$ constant

\[
v_{t, \text{upstream}} = v_{t, \text{downstream}}
\]

\[
v \cos \mu = (v + dv) \cos (\mu + dv)
\]

\[
dv \to 0
\]

\[
v \cos \mu = v \cos \mu - dv \sin \mu + dv \cos \mu - dv \sin \mu
\]

\[
\sin \mu = 1/M, \quad \sin^2 \mu + \cos^2 \mu = 1
\]

\[
\frac{dv}{v} = \frac{1}{\sqrt{M^2 - 1}} dv
\]

(VIII.1)

Relation Between $M$ and $dv$

- Relate $v$ and $M$

\[
v = Ma
\]

\[
\frac{dv}{v} = \frac{dM}{M} + \frac{da}{a}
\]

\[
\frac{dv}{v} = \frac{dM}{M} + \frac{d\sqrt{T}}{\sqrt{T}} = \frac{dM}{M} + \frac{1}{2} \frac{dT}{T}
\]

\[
\frac{dv}{v} = \frac{dM}{M} \left[ \frac{(\gamma - 1)/2}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \right] = \frac{dM}{M} \left[ \frac{1}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \right]
\]

Energy Conservation

\[
T = T' \left(1 + \frac{\gamma - 1}{2} M^2\right) = \text{const.}
\]

\[
\frac{dT}{T} = 0 = \frac{dT}{T} + \frac{1}{2} \frac{dM}{M} \left[ \frac{(\gamma - 1)/2}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \right]
\]

\[
\frac{dT}{T} = \frac{1}{2} \frac{dM}{M} \left[ \frac{1}{1 + \left(\frac{\gamma - 1}{2}\right) M^2} \right]
\]

\[
\frac{dT}{T} = \frac{(\gamma - 1)/2}{1 + \left(\frac{\gamma - 1}{2}\right) M^2}
\]
**Relation Between M and \( dv \) (con’t)**

- Relate VIII.1 and last eqn.

\[
\frac{dv}{v} = \frac{1}{\sqrt{M^2 - 1}} \frac{dM}{M} = \frac{1}{\left(1 + \frac{\gamma - 1}{2} M^2\right)} M \frac{dM}{M} \]

\[
dv = \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM}{M} \quad \text{(VIII.2)}
\]

- \( dM \) is change in Mach number associated with \( dv \) turn angle

- Need finite angle, \( \delta = v_2 - v_1 \) and finite \( \Delta M \)

\[
v_2 - v_1 = \int_{v_1}^{v_2} dv = \int_{M_1}^{M_2} \frac{\sqrt{M^2 - 1}}{M} \frac{dM}{M_1 + \frac{\gamma - 1}{2} M^2} \]

- Perform Integration

\[
v_2 - v_1 = \delta = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \left[ \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} \right) - \tan^{-1} \left( \frac{\sqrt{M_2^2 - 1}}{M_{M_2}} \right) \right] \]

\[
\text{(VIII.3)}
\]

- So, given \( \delta \) (=\( v_2 - v_1 \)) and \( M_1 \)
  - could “solve” VIII.3 for \( M_2 \)
- Can not invert VIII.3 analytically (\( M_2 = f(M_1, \delta) \))
  - either use *iterative* (e.g., numerical or guessing) method
  - or find \( v \) as a function of \( M \) and tabularize or graph solution
Tabular Solutions/Reference Condition

\[ \nu_2 - \nu_1 = \left[ \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \frac{\gamma - 1}{\gamma + 1} (M_1^2 - 1) - \tan^{-1} \sqrt{M_1^2 - 1} \right] M_1 \]

- Want to find \( \nu = \nu(M) \) [really \( \nu_2 = \nu_2(M_2) \)] for any \( M \)
  - need to choose (arbitrary) reference condition, i.e., pick an \( M \) where \( \nu = 0 \)
  - let's choose \( \nu = 0 \) at \( M = 1 \)

\[ \nu = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \frac{\gamma - 1}{\gamma + 1} (M_1^2 - 1) - \tan^{-1} \sqrt{M_1^2 - 1} \]

(\text{VIII.4} \text{ Appendix D (John)} \text{ for } \gamma = 1.4, M \leq 5)

- \( \nu \) represents angle through which a sonic flow would have to turn to reach \( M \)

\[ \nu \text{ is turn angle for } M = 1 \text{ to } M_1 \]

Using the Prandtl Meyer Tables

- To find \( M_2 \) given \( M_1 \) and \( \delta \)
  - find \( \nu_1 \) (for given \( M_1 \)) from table
  - get \( \nu_2 \) from \( \delta = \nu_2 - \nu_1 \)
  - look up \( \nu_2 \) in table to find \( M_2 \)

- To find \( T_2, p_2, \ldots \)
  - use isentropic flow relations since expansion is isentropic (no shock)
    - e.g.,
      \[ \frac{T_2}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 \implies \frac{T_2}{T_1} = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2} \]
Example

- **Given:** Uniform Mach 2 flow of nitrogen at 300 K flows over compound wall corner: two turns, 20° and 6°

- **Find:**
  M and T after final turn

- **Assume:** N₂ is TPG/CPG with γ=1.4, steady, adiabatic, no work, inviscid,....

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**Example (con’t)**

- **Analysis:** (class exercise)
  1. To find \(M_2\) given \(M_1\) and \(\delta\)
     a. find \(\nu_1\) (for given \(M_1\)) from table
     b. get \(\nu_2\) from \(\delta=\nu_2-\nu_1\)
     c. look up \(\nu_2\) in table to find \(M_2\)

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Example (con’t)

• Analysis: (solution)

Prandtl Meyer Fan Angle

• Fan angle
  – angle between first and last Mach wave
  – useful to determine when expansion has ended in flowfield for a given distance away from wall
  
• From geometry

\[
\text{Fan Angle} = \mu_1 - (\mu_2 - \delta) = (\mu_1 - \mu_2) + \delta \quad (\text{VIII.5})
\]
Prandtl Meyer Turns at High M

- Examine plot of $\nu$ as function of $M$

$$\nu_{\text{max}} = 130.45^\circ$$

- As $M$ increases, reach maximum turn angle ($\nu_{\text{max}} \approx 130.5^\circ$ for $\gamma = 1.4$)
- So as $M_1$ increases, max. angle flow can turn ($\delta_{\text{max}}$) decreases

Example
$M_1 = 2 \Rightarrow \delta_{\text{max}} = 130.45^\circ - 26.38^\circ = 104.1^\circ$
$M_1 = 6 \Rightarrow \delta_{\text{max}} = 130.45^\circ - 84.96^\circ = 45.5^\circ$

Maximum Turn Angle

- Analytic Expression

$$\nu = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} M^2 - 1 \right) - \tan^{-1} \sqrt{M^2 - 1}$$

For $M \to \infty: \sqrt{M; tan^{-1}(M)} \to 90^\circ$

$$\nu_{\text{max}} = \left( \frac{\gamma + 1}{\gamma - 1} - 1 \right) \times 90^\circ$$ (VIII.6)

- As $\gamma$ decreases (higher temperatures, bigger molecules), maximum turn angle increases
- $\delta_{\text{max}}$ smaller in real flows
  - $T$ and $p$ drop through turn
    - Condensation of gas
    - Nonequilibrium flow
Continuous (Smooth) Expansions

- Already showed that it does not matter if expansion turn is sharp or smooth
  - still get same solution,
    - P-M fan=infinite set of Mach waves
  - unless we exceed the maximum turning angle, final properties just function of total turn angle
  - smooth turn just means expansion process takes place over longer distance

Continuous (Smooth) Compressions

- What happens if we have a smooth concave turn?
  - since flow direction change is small, can still get set of weak Mach waves
  - however unlike expansions, compressions merge
  - together they coalesce to form oblique shock
  - flow that went through PM compression is isentropic, outer flow has entropy rise (p_o loss)
  - size of PM region depends on M_1 and curvature