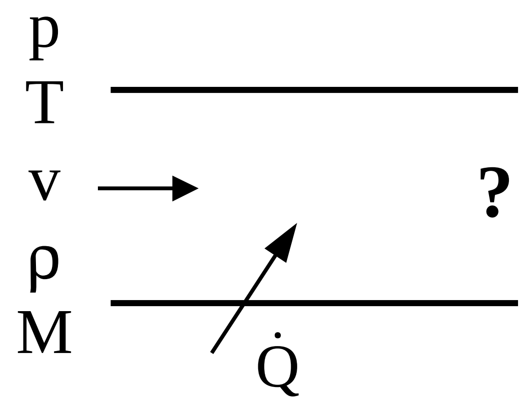
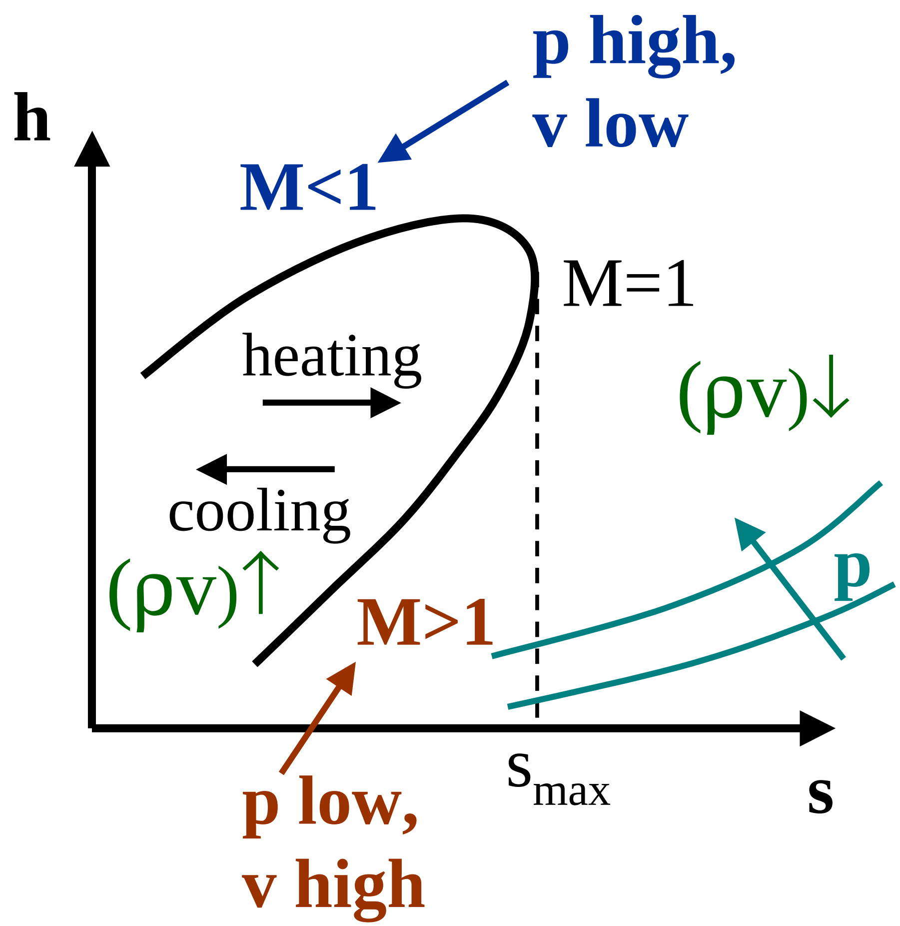
Rayleigh Flow - Thermodynamics



- Steady, 1-d, constant area, inviscid flow with no external work but *with reversible heat transfer (heating or cooling)*
- Conserved quantities (mass, momentum eqs.)
  - since A=const: mass flux=**ρv=constant≡G**
  - since no forces but pressure: **$p+\rho v^2$=constant**
  - combining: **$p+G^2/\rho$=constant**
- On h-s diagram, can draw this **Rayleigh Line**

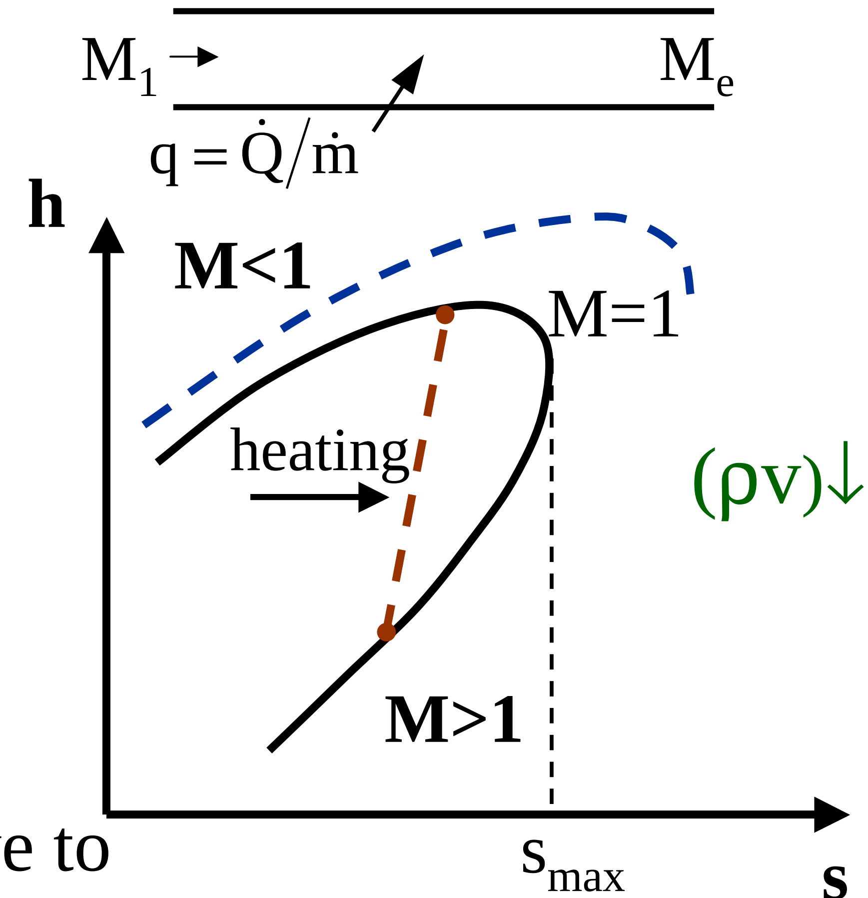
Rayleigh Line

- **$p+\rho v^2$=constant**,
  ⇒high p results in low v
- ds=δQ/T (reversible)
  - heating increases s
  - cooling decreases s
  - M=1 at maximum s
- Change mass flux
  new Rayleigh line



Rayleigh Line - Choking

- Heat addition can only increase entropy
- For **enough heating**, M_e=1 (Q_{in}=Q_{max})
  - heat addition can lead to choked flow
- What happens if Q_{in}>Q_{max}
  - **subsonic flow**: must move to different Rayleigh line (– – –), i.e., lower mass flux
  - **supersonic flow**: get a shock (– – –)



stays on same Rayleigh line: ρv and p+ρv² also const. for normal shock

Differential Mach Equations

- Simplify (X.4-5) for *f*=dA=0

$$\frac{dM^2}{M^2} = \frac{\left(1+\gamma M^2\right)\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\frac{\delta q}{c_p T_o} \quad \textbf{(X.21)}$$

$$\frac{dT_o}{T_o} = \frac{\delta q}{c_p T_o} \quad \textbf{(X.25)}$$

$$\frac{dp}{p} = \frac{-\gamma M^2}{1+\gamma M^2}\frac{dM^2}{M^2} \quad \textbf{(X.22)}$$

$$\frac{dp_o}{p_o} = -\frac{\gamma}{2}M^2\frac{dT_o}{T_o} \quad \textbf{(X.26)}$$

$$\frac{dT}{T} = \frac{dh}{h} = \frac{1-\gamma M^2}{1+\gamma M^2}\frac{dM^2}{M^2} \quad \textbf{(X.23)}$$

$$\frac{ds}{c_p} = \frac{1-M^2}{1+\gamma M^2}\frac{dM^2}{M^2} \quad \textbf{(X.27)}$$

$$\frac{dv}{v} = -\frac{d\rho}{\rho} = \frac{-1}{1+\gamma M^2}\frac{dM^2}{M^2} \quad \textbf{(X.24)}$$

- **sign changes**

Property Variations

- What can change sign in previous equations
  - $(1\text{-}M^2)$ and δq
  - $(1\text{-}\gamma M^2)$ in dh,dT for M<1

- s, T_o just like q
- p_o opposite of T_o and **s**
- Heating: M→1; cooling opposite
- p, ρ opposite of v ($p+ \rho v^2$=const)
- h,T same as p,ρ unless $M^2<1/\gamma$ (low speed flow: T oppos. p,ρ)

δq	**M<1** <0	**M<1** >0	**M>1** <0	**M>1** >0
s, T_o	↓	↑	↓	↑
p_o	↑	↓	↑	↓
M, v	↓	↑	↑	↓
p, ρ	↑	↓	↓	↑
h, T			↓	↑
$M^2<\gamma^{-1}$	↓	↑		
$M^2>\gamma^{-1}$	↑	↓		

δq<0⇒cooling
δq>0⇒heating

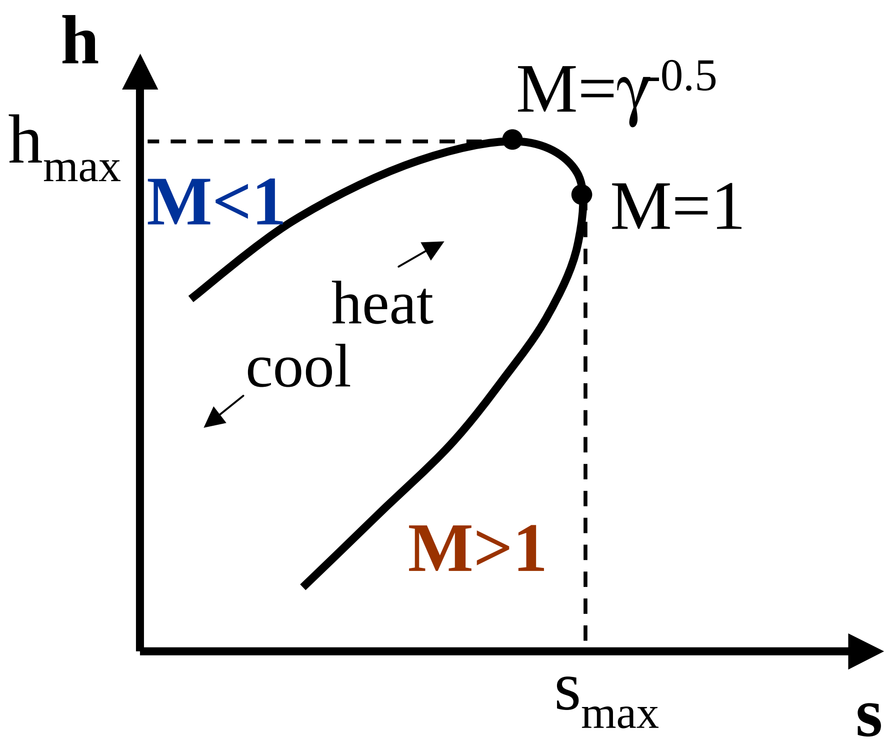
Rayleigh Line Extremum

- Combine X.23 and X.27

$$\frac{dh}{ds} = \frac{h}{c_p}\frac{1-\gamma M^2}{1-M^2}$$

$$\frac{dh}{ds} = 0 \text{ for } M = \frac{1}{\sqrt{\gamma}}$$

$$\frac{dh}{ds} = \infty \text{ for } M = 1$$



Integrated Mach Equations

- Integrating (X.22-27)

$$\frac{dT_o}{T_o} = \frac{\delta q}{c_p T_o} \Rightarrow \int dT_o = \int \frac{\delta q}{c_p} \Rightarrow \left(T_{o2} - T_{o1}\right) = \frac{q_{12}}{c_p} \quad \textbf{(X.28)} \text{ (energy)}$$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \quad \textbf{(X.29)}$$

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \quad \textbf{(X.30)}$$

$$\frac{\text{v}_1}{\text{v}_2} = \frac{\rho_2}{\rho_1} = \frac{1+\gamma M_2^2}{1+\gamma M_1^2} \left(\frac{M_1}{M_2}\right)^2 \quad \textbf{(X.31)}$$

$$\frac{p_{o2}}{p_{o1}} = \frac{p_2 \frac{p_{o2}}{p_2}}{p_1 \frac{p_{o1}}{p_1}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}} \quad \textbf{(X.32)}$$

$$\frac{T_{o2}}{T_{o1}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right) \quad \textbf{(X.33)}$$

$$\frac{s_2 - s_1}{c_p} = \ln\left[\left(\frac{M_2}{M_1}\right)^2 \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^{(\gamma+1)/\gamma}\right] \quad \textbf{(X.34)}$$

Solution Approaches

- Two methods to "solve" Rayleigh problems, i.e., for state change from $1 \rightarrow 2$
- In both, need to "know" one state (including *M*) and something about other state (1 TD property or *q* or *M*)
- **Method 1**: direct solution of integrated equations (X.28-34)
  - requires iterative solution
- **Method 2**: tabular solutions using reference condition
  - uses sonic reference condition (similar to A/A* and Fanno methods)
  - sonic condition based on heat transfer required for flow to go sonic; $M_2=1$, $T_{o2}= T_o^*$

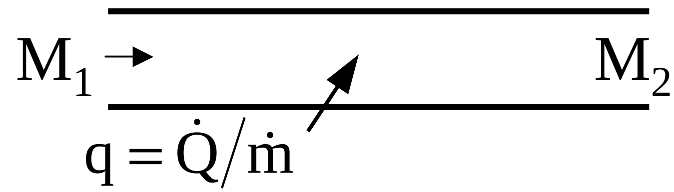
Direct Solution

- Use two equations to find M change
  - e.g., if state 1 and q_{12} known use X.28 and X.33

a) get T_o ratio from X.28

$$\frac{T_{o2}}{T_{o1}} = 1 + \frac{q_{12}}{c_p T_{o1}}$$

doesn't matter how q changes along length, only total q required



b) get M_2 from X.33 (must know M_1)

$$\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right) = \frac{T_{o2}}{T_{o1}}$$

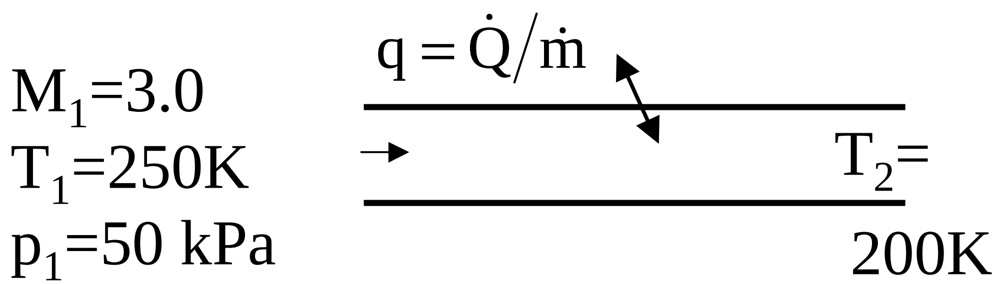
requires iterative solution for M_2

c) get rest of property changes M_2 from X.29-34, e.g., from X.29

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

Cooled Duct Example

- **Given:** Nitrogen flowing in constant area duct, enters at M=3, T=250 K, p=50 kPa and exits with temperature of 200 K

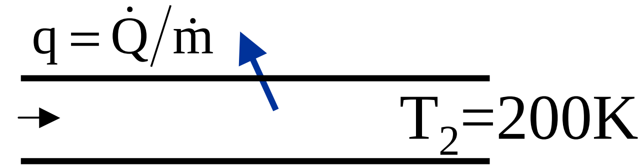


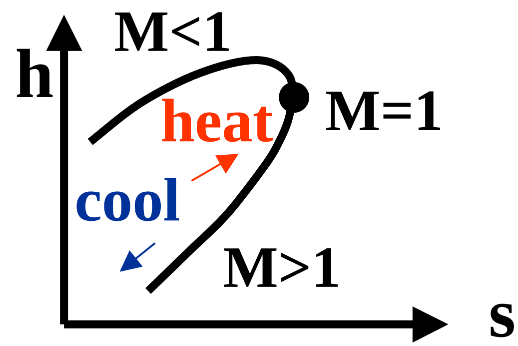
- **Find:**
  1. M_2
  2. *q* (including direction)

- **Assume:** N_2 is tpg/cpg, γ=1.4, steady, reversible, no work

Cooled Duct - Solution

M_1=3.0
T_1=250K
p_1=50 kPa





- **Analysis:** (from X.30)
  - ***M*₂**

$$\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2\left(\frac{M_2}{M_1}\right)^2=\frac{T_2}{T_1}$$

$$\left(\frac{1+1.4(3)^2}{1+1.4M_2^2}\right)^2\left(\frac{M_2}{3}\right)^2=\frac{200}{250}=0.80$$

$$\Rightarrow M_2=0.21,3.41$$

since started M>1, can't be subsonic

  - ***q*** (energy: ins = outs)

$$q=c_p\left(T_{o1}-T_{o2}\right)$$

$$=\frac{1.4}{1.4-1}\,\frac{8314}{28}\,\frac{J/kmolK}{kg/kmol}(700-665)K$$

$$=36\,kJ/kg$$

$$T_{o1}=T_1\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$=250K\left(1+\frac{1.4-1}{2}9^2\right)=700K$$

$$T_{o2}=200K\left(1+0.2(3.41)^2\right)=665K$$

Reference Solution Method

- Reference state has M=1, $T \equiv T^*$, $p \equiv p^*$,...

⇒sonic, but not same state as A or Fanno

- State 2→sonic in integrated equations (X.29-34)

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2} \quad \text{(X.35)}$$

$$\frac{T}{T^*} = M^2 \left(\frac{1+\gamma}{1+\gamma M^2} \right)^2 \quad \text{(X.36)}$$

$$\frac{v}{v^*} = \frac{\rho^*}{\rho} = \frac{1+\gamma}{1+\gamma M^2} M^2 \quad \text{(X.37)}$$

(X.38)

$$\frac{p_o}{p_o^*} = \frac{1+\gamma}{1+\gamma M^2} \left(\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}} \right)^{\frac{\gamma}{\gamma-1}}$$

(X.39)

$$\frac{T_o}{T_o^*} = \left(\frac{1+\gamma}{1+\gamma M^2} \right)^2 M^2 \frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}$$

(X.40)

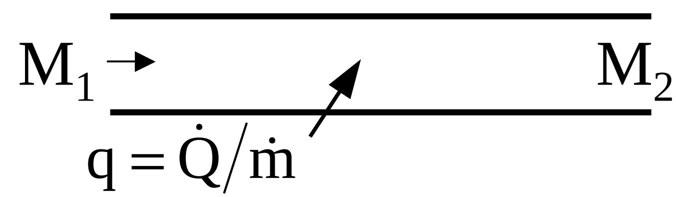
$$\frac{s-s^*}{c_p} = ln\left\{ M^2 \left(\frac{1+\gamma}{1+\gamma M^2} \right)^{\frac{1+\gamma}{\gamma}} \right\}$$

- **Note: *prop/prop** = f(*M*) only**
- Values for γ=1.4 in Appendix F, John

Use of Tables

- To get change in M, use change in property ratio (like using A/A*), e.g.,

$$\frac{T_{o2}}{T_{o1}} = \frac{T_{o2}/T_o^*}{T_{o1}/T_o^*} = \frac{\left. T_o/T_o^* \right|_{M_2}}{\left. T_o/T_o^* \right|_{M_1}}$$



- For example, if state 1 and q_{12} known

1) again get T_o ratio from X.28

$$\frac{T_{o2}}{T_{o1}} = 1 + \frac{q_{12}}{c_p T_{o1}}$$

2) look up T_o/T_o^* at M_1

3) calculate T_o/T_o^* at M_2

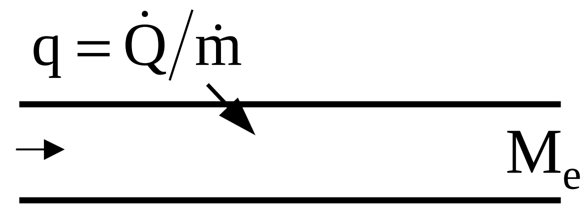
$$\frac{T_{o2}}{T_o^*} = \frac{T_{o2}}{T_{o1}} \times \frac{T_{o1}}{T_o^*}$$

4) look up M_2 that corresponds to calculated T_o/T_o^*

Heated Duct Example

- **Given:** Air flowing in constant area duct, enters at M=0.2 and given inlet conditions and a heating rate of 765 kJ/kg

M_i=0.2
T_{oi}=300K
p_{oi}=600 kPa

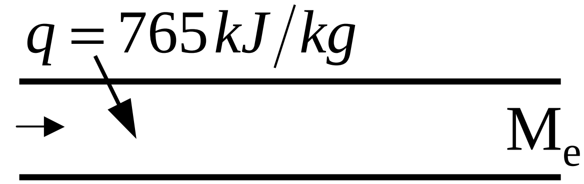


- **Find:**
  1. M_e, p_{oe}, T_e
  2. q_{max} for M_e=1

- **Assume:** air is tpg/cpg, γ=1.4, steady, reversible, no work

Heated Duct - Solution

M_i=0.2
T_{oi}=300K
p_{oi}=600 kPa



M	To/To*	p_o/p_o*
0.19	0.15814	1.23765
0.20	0.17355	1.23460
0.21	0.18943	1.23142
0.44	0.59748	1.13936
0.45	0.61393	1.13508
0.46	0.63007	1.13082
3.53	0.61393	5.47054

(Appendix F)

- **Analysis:**
  - **M_e**

(X.28, energy)

$$\frac{T_{oe}}{T_{oi}} = 1 + \frac{q}{c_p T_{oi}}$$

$$= 1 + \frac{765 \times 10^3\, J/kg}{1004 \frac{J}{kgK}(300K)} = 3.54$$

$$\frac{T_{oe}}{T_o^*} = \frac{T_{oe}}{T_{oi}} \frac{T_{oi}}{T_o^*}\bigg|_{M=0.2} = 3.54(0.17355) = 0.614 \Rightarrow M_e = 0.45$$

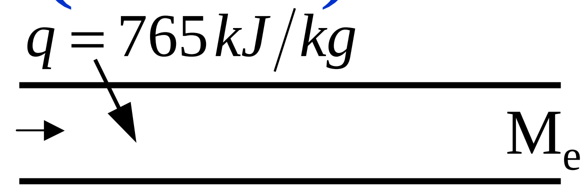
another solution for M=3.53, but since started with M<1, can't be supersonic

  - **p_{oe}**

$$p_{oe} = \frac{p_{oe}}{p_o^*} \frac{p_o^*}{p_{oi}} p_{oi} = 1.1351 \frac{1}{1.2346} 600kPa = 552kPa$$

Heated Duct Solution (con't)

M_i=0.2
T_{oi}=300K
p_{oi}=600 kPa



– **T_e**

$$T_e = \frac{T_e}{T_{oe}}\frac{T_{oe}}{T_{oi}}T_{oi}$$

$$T_e = \frac{1}{1+\frac{0.4}{2}0.45^2}3.54(300K) = 1021K$$

– **q_{max}** requires choked exit, $T_{oe}=T^*$

$$q_{max} = c_p\left(T_o^* - T_o\right) = c_pT_{o1}\left(\frac{T_o^*}{T_{o1}} - 1\right)$$

$$= 1004\frac{J}{kgK}300K\left(\frac{1}{0.17355} - 1\right) = 1.43\frac{MJ}{kg}$$

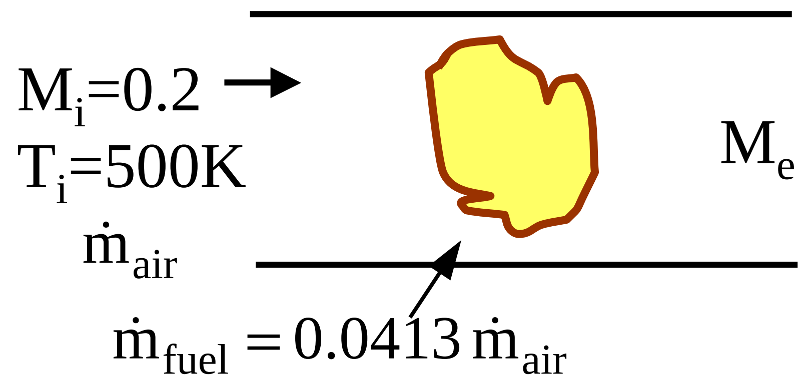
$$T_o^* = \frac{300K}{0.17355} = 1728K$$

$$T^* = T_o^* \Big/ \left(1+\frac{\gamma-1}{2}\right) = 1440K$$

Combustor Example

- **Given:** Air entering constant combustor, enters at M=0.2 and given inlet conditions.
  q=heating value of fuel (45.5 MJ/kg$_{fuel}$) $\times \dfrac{\dot{m}_{fuel}}{\dot{m}_{air}}$

- **Find:**
  1. T_e and compare to value you would calculate if neglected kinetic energy change
  2. p_o loss



- **Assume:** air is tpg/cpg, γ=1.4, steady, reversible, no work,
  $$\frac{\dot{m}_{fuel}}{\dot{m}_{air}} \ll 1 \text{ (can neglect mass addition, just use q)}$$

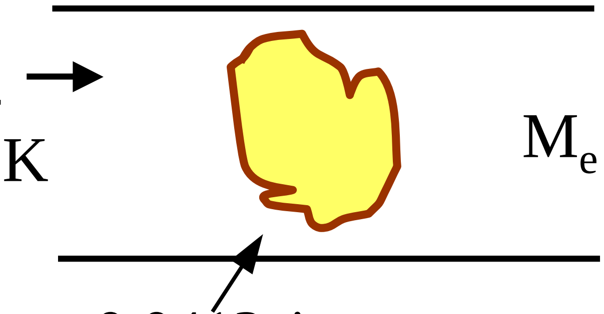
Combustor Solution

- **Analysis:** (energy)
  - T_e

$$\frac{T_{oe}}{T_{oi}} = 1 + \frac{q}{c_p T_{oi}}$$

? combustion just another kind of heat addition

$M_i=0.2$ →
$T_i=500K$
$\dot{m}_{air}$



M_e

$\dot{m}_{fuel} = 0.0413\,\dot{m}_{air}$

$$q = 0.0413\frac{kg_{fuel}}{kg_{air}}\left(45.5\frac{MJ}{kg_{fuel}}\right) = 1.88\frac{MJ}{kg_{air}}$$

$$T_{oi} = T_i\left(1 + \frac{\gamma-1}{2}M^2\right) = 500K(1.008) = 504K$$

$$T_{oe} = T_{oi}\left(1 + \frac{1.88\times10^6}{1004(504)}\right) = 504K(4.715) = 2376K$$

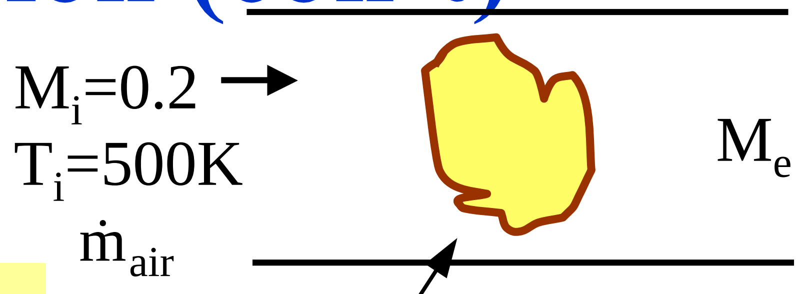
even for this subsonic flow, kinetic energy reduces final T by 160K

(Appendix F)

$$\frac{T_{oe}}{T_o^*} = \frac{T_{oe}}{T_{oi}}\frac{T_{oi}}{T_o^*} = 4.715(0.1736) = 0.818 \Rightarrow M_e = 0.60$$

$$T_e = \frac{T_{oe}}{1+M^2(\gamma-1)/2} = \frac{2376K}{1+0.2(0.6)^2} = 2216K$$

Combustor Solution (con't)



$$\dot{m}_{fuel} = 0.0413\, \dot{m}_{air}$$

– p_{oe}/p_{oi}

$$\frac{p_{oe}}{p_{oi}} = \frac{p_{oe}}{p_o^*}\frac{p_o^*}{p_{oi}} = 1.0753\frac{1}{1.2346} = 0.871$$

(Appendix F)

- So heat addition (burning fuel) in an engine gives p_o loss as was case with friction
- As we lose p_o (~13% loss here)
  - less thrust
  - less work output

Choking

- Heating of a flow can lead to choking just like friction
- For supersonic flow, depending on back pressure, can get
  - – underexpanded flow
  - – overexpanded flow
  - – shock inside duct