Rayleigh Flow - Thermodynamics

- Steady, 1-d, constant area, inviscid flow with no external work but with reversible heat transfer (heating or cooling)
- Conserved quantities (mass, momentum eqs.)
  - since A=const: mass flux=\( \rho v = \text{constant} \equiv G \)
  - since no forces but pressure: \( p + \rho v^2 = \text{constant} \)
  - combining: \( p + G^2/\rho = \text{constant} \)
- On h-s diagram, can draw this Rayleigh Line

Rayleigh Line

- \( p + \rho v^2 = \text{constant} \), \( \Rightarrow \) high \( p \) results in low \( v \)
- \( ds = \delta Q/T \) (reversible)
  - heating increases \( s \)
  - cooling decreases \( s \)
  - \( M = 1 \) at maximum \( s \)
- Change mass flux new Rayleigh line
Rayleigh Line - Choking

- Heat addition can only increase entropy
- For enough heating, \( M_e = 1 \) (\( Q_{in} = Q_{max} \))
  - heat addition can lead to choked flow
- What happens if \( Q_{in} > Q_{max} \)
  - subsonic flow: must move to different Rayleigh line (\( \longrightarrow \)), i.e., lower mass flux
  - supersonic flow: get a shock (\( \longrightarrow \))
    - stays on same Rayleigh line: \( \rho v \) and \( p + \rho v^2 \) also const. for normal shock

Differential Mach Equations

- Simplify (X.4-5) for \( f = dA = 0 \)

\[
\begin{align*}
\frac{dM^2}{M^2} &= \left( \frac{1 + \gamma M^2}{1 - M^2} \right) \frac{\delta q}{c_p T_o} \\
\frac{dp}{p} &= -\frac{\gamma M^2}{1 + \gamma M^2} \frac{dM^2}{M^2} \\
\frac{dT}{T} &= \frac{\delta h}{h} = \frac{1 - \gamma M^2}{1 + \gamma M^2} \frac{dM^2}{M^2} \\
\frac{dv}{v} &= -\frac{dp}{\rho} = \frac{-1}{1 + \gamma M^2} \frac{dM^2}{M^2}
\end{align*}
\]

- sign changes
Property Variations

- What can change sign in previous equations
  - \((1-M^2)\) and \(\delta q\)
  - \((1-\gamma M^2)\) in \(dh,dT\) for \(M<1\)

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<tr>
<th>(\delta q)</th>
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- \(s, T_o\) just like \(q\)
- \(p_o\) opposite of \(T_o\) and \(s\)
- Heating: \(M\to1\); cooling opposite
- \(p, \rho\) opposite of \(v\) (\(p+\rho v^2=\text{const}\))
- \(h, T\) same as \(p, \rho\) unless \(M^2<1/\gamma\)
  (low speed flow: \(T\) oppos. \(p, \rho\))

Rayleigh Line Extremum

- Combine X.23 and X.27

\[
\frac{dh}{ds} = \frac{h}{c_p} \frac{1-\gamma M^2}{1-M^2}
\]
\[
\frac{dh}{ds} = 0 \text{ for } M = \frac{1}{\sqrt{\gamma}}
\]
\[
\frac{dh}{ds} = \infty \text{ for } M = 1
\]
**Integrated Mach Equations**

- Integrating (X.22-27)
  \[
  \frac{dT_o}{T_o} = \frac{\delta q}{c_p T_o} \Rightarrow \int \frac{dT_o}{T_o} = \int \frac{\delta q}{c_p} \Rightarrow \frac{T_{o2} - T_{o1}}{T_{o1}} = \frac{q_{12}}{c_p} \quad \text{(energy)} \quad \text{(X.28)}
  \]

  \[
  \begin{align*}
  \frac{p_2}{p_1} &= \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \quad \text{(X.29)} \\
  \frac{T_2}{T_1} &= \left(1 + \gamma M_2^2 \right)^2 \left(\frac{M_2}{M_1}\right)^2 \quad \text{(X.30)} \\
  \frac{\rho_2}{\rho_1} &= \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \left(\frac{M_1}{M_2}\right)^2 \quad \text{(X.31)}
  \end{align*}
  \]

- \[
  \frac{p_{o2}}{p_{o1}} = \frac{p_2}{p_1} \frac{p_{o2}}{p_{o1}} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \left(\frac{1 + \gamma - 1 M_2^2}{1 + \gamma - 1 M_1^2}\right)^{\frac{7}{2}} \quad \text{(X.32)}
  \]

- \[
  \frac{T_{o2}}{T_{o1}} = \left(1 + \gamma M_1^2 \right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(1 + \gamma - \frac{1}{2} M_1^2 \right) \left(1 + \gamma - \frac{1}{2} M_2^2 \right) \quad \text{(X.33)}
  \]

- \[
  s_2 - s_1 = \frac{\ln \left(\frac{M_2}{M_1}\right)^2 \left(1 + \gamma M_2^2 \right)^{\frac{\gamma + 1}{\gamma}}}{c_p} \quad \text{(X.34)}
  \]

**Solution Approaches**

- Two methods to “solve” Rayleigh problems, i.e., for state change from 1→2
- In both, need to “know” one state (including $M$) and something about other state (1 TD property or $q$ or $M$)
- **Method 1**: direct solution of integrated equations (X.28-34)
  – requires iterative solution
- **Method 2**: tabular solutions using reference condition
  – uses sonic reference condition (similar to A/A* and Fanno methods)
  – sonic condition based on heat transfer required for flow to go sonic: $M_2 = 1$, $T_{e2} = T_{o*}$
Direct Solution

- Use two equations to find $M$ change
  - e.g., if state 1 and $q_{12}$ known use X.28 and X.33
  
  a) get $T_o$ ratio from X.28
  
  $$\frac{T_o_2}{T_o_1} = 1 + \frac{q_{12}}{c_p T_o_1}$$
  
  doesn’t matter how $q$ changes along length, only total $q$ required

  b) get $M_2$ from X.33 (must know $M_1$)
  
  $$\left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)^{\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}} = \frac{T_o_2}{T_o_1} \text{ requires iterative solution for } M_2$$

  c) get rest of property changes $M_2$ from X.29-34, e.g., from X.29
  
  $$\frac{p_2}{p_1} = 1 + \gamma M_1^2$$

Cooled Duct Example

- **Given:** Nitrogen flowing in constant area duct, enters at $M=3$, $T=250 \text{ K}$, $p=50 \text{ kPa}$ and exits with temperature of 200 K

- **Find:**
  1. $M_2$
  2. $q$ (including direction)

- **Assume:** $N_2$ is tpg/cpg, $\gamma=1.4$, steady, reversible, no work
Cooled Duct - Solution

- Analysis: (from X.30)

\[ M_2 = \frac{\left(1 + \gamma M_1^2 \right)^{\frac{2}{3}}}{\left(1 + \gamma M_2^2 \right)^{\frac{2}{3}}} \]

\[ \frac{(1 + 1.4(3)^2) M_2}{(1 + 1.4 M_2^2) M_1} \]

\[ M_2 = 0.21, 3.41 \] since started M>1, can’t be subsonic

- \( q \) (energy: ins = outs)

\[ q = c_p(T_{o1} - T_{o2}) \]

\[ = \frac{1.4}{1.4 - 1} \times 8314 \text{ J/kmolK} \times (700 - 665) \text{K} \]

\[ = 36 \text{ kJ/kg} \]

\[ T_{o1} = T_1 \left(1 + \frac{\gamma - 1}{2} M_2^2 \right) \]

\[ = 250K \left(1 + \frac{1.4 - 1 \times 2}{2} \right) = 700K \]

\[ T_{o2} = 200K \left(1 + 0.2(3.41)^2 \right) = 665K \]

Reference Solution Method

- Reference state has \( M=1, T=T^*, p=p^* \ldots \) \( \Rightarrow \) sonic, but not same state as \( A^* \) or Fanno

\[ \frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2} \]

\[ (X.35) \]

\[ \frac{T}{T^*} = M^2 \left(\frac{1 + \gamma}{1 + \gamma M_2^2}\right)^2 \]

\[ (X.36) \]

\[ \frac{v}{v^*} = \frac{\rho}{\rho^*} = \frac{1 + \gamma}{1 + \gamma M_2^2} M^2 \]

\[ (X.37) \]

\[ \text{Note: prop/prop}^* = f(M) \text{ only} \]

- Values for \( \gamma=1.4 \) in Appendix F, John
Use of Tables

- To get change in $M$, use change in property ratio (like using $A/A^*$), e.g.,

\[
\frac{T_o^2}{T_o} = \frac{T_o^2}{T_o^*} = \frac{T_o}{T_o^*}
\]

- For example, if state 1 and $q_{12}$ known
  1) again get $T_o$ ratio from X.28
  \[
  \frac{T_o^2}{T_o} = 1 + \frac{q_{12}}{c_p T_o}
  \]
  2) look up $T_o/T_o^*$ at $M_1$
  3) calculate $T_o/T_o^*$ at $M_2$
  \[
  \frac{T_o^2}{T_o} = \frac{T_o^2}{T_o} \times \frac{T_o}{T_o^*}
  \]
  4) look up $M_2$ that corresponds to calculated $T_o/T_o^*$

Heated Duct Example

- **Given:** Air flowing in constant area duct, enters at $M=0.2$ and given inlet conditions and a heating rate of 765 kJ/kg

- **Find:**
  1. $M_e$, $p_{oe}$, $T_e$
  2. $q_{max}$ for $M_e=1$

- **Assume:** air is tpg/cpg, $\gamma=1.4$, steady, reversible, no work
Heated Duct - Solution

- Analysis:
  \[ M_e = 0.2 \]
  \[ T_{oi} = 300K \]
  \[ p_{oi} = 600 \text{ kPa} \]
  \[ q = 765 \text{ kJ/kg} \]
  \[ \frac{p_{oe}}{p_{oi}} = 1.1351 \]
  \[ T_{oe} = T_{oi} \]

(X.28, energy)

\[ T_{oe} = T_{oi} + c_p T_{oi} \frac{q}{1004 J/(kgK)} = 3.54 (300K) \]

\[ T_{oe} = 1 + \frac{q}{c_p T_{oi}} = 1 + \frac{765 \times 10^3 J/kg}{1004 J/(kgK)} = 3.54 \]

\[ M_e = \frac{T_{oe}}{T_{oi}^*} \]

\[ p_{oe} = 1.1351 \times 600kPa = 552kPa \]

\( \text{ another solution for } M=3.53, \text{ but since } M<1, \text{ can’t be supersonic } \)

\( \text{ requires choked exit, } T_{oe} = T_{*} \)

Heated Duct Solution (con’t)

- \( T_e \)
  \[ T_e = T_{oe} \]
  \[ T_{oe} = T_{oi} \]
  \[ T_{e} = \frac{1}{1 + \frac{0.4}{0.45^2}} \]

\[ T_e = 3.54(300K) = 1021K \]

- \( q_{max} \)
  \[ q_{max} = c_p \left( T_{*} - T_{o} \right) = c_p T_{oi} \left( \frac{T_{*}}{T_{oi}} - 1 \right) \]

\[ q_{max} = 1004 J/(kgK) \times 300K \left( \frac{1}{0.17355} - 1 \right) = 1440K \]

\[ T_{*} = T_{o} \left( 1 + \frac{\gamma - 1}{2} \right) = 1728K \]

\[ T_{o} = 0.17355 \]

\[ M_e = 0.2 \]
Combustor Example

- **Given:** Air entering constant combustor, enters at \( M = 0.2 \) and given inlet conditions.
  \( q \) = heating value of fuel (45.5 MJ/kg\(_{\text{fuel}}\)) \( \times \frac{m_{\text{fuel}}}{m_{\text{air}}} \)

- **Find:**
  1. \( T_e \) and compare to value you would calculate if neglected kinetic energy change
  2. \( p_o \) loss

- **Assume:** air is tpg/cpg, \( \gamma = 1.4 \), steady, reversible, no work, \( \frac{m_{\text{fuel}}}{m_{\text{air}}} << 1 \) (can neglect mass addition, just use \( q \))

Combustor Solution

- **Analysis:** (energy)
  \( T_e \)
  \( \frac{T_{oe}}{T_{oi}} = 1 + \frac{q}{C_p T_{oi}} \) \( \frac{45.5 \text{ MJ}}{1.88 \text{ MJ/kg}} = 24 \) combustion just another kind of heat addition
  \( q = 0.0413 \frac{\text{kg}_{\text{fuel}}}{\text{kg}_{\text{air}}} \times 45.5 \text{ MJ/kg}_{\text{fuel}} = 1.88 \frac{\text{MJ}}{\text{kg}_{\text{air}}} \)
  \( T_{oi} = T_i \left( 1 + \frac{\gamma - 1}{2} M^2 \right) = 500K(1.008) = 504K \)
  \( T_{oe} = T_{oi} \left( 1 + \frac{1.88 \times 10^6}{1004(504)} \right) = 504K \left( 4.715 \right) = 2376K \) (Appendix F)
  \( \frac{T_{oe}}{T_o} \)
  \( \frac{T_{oe}}{T_{oi}} = 4.715(0.1736) = 0.818 \Rightarrow M_e = 0.60 \)
  \( \frac{T_{oe}}{T_o} \)  
  \( T_e = \frac{2376K}{1 + M^2 (\gamma - 1)/2} = \frac{2376K}{1 + 0.2(0.6)^2} = 2216K \) even for this subsonic flow, kinetic energy reduces final \( T \) by 160K
Combustor Solution (con’t)

\[\frac{p_{oe}}{p_{oi}} = \frac{p_{oe}}{p_{o}} \cdot \frac{p_{o}}{p_{oi}} = 1.0753 \cdot \frac{1}{1.2346} = 0.871\]

\[\dot{m}_{\text{air}} = 0.0413 \dot{m}_{\text{air}}\]

- So heat addition (burning fuel) in an engine gives \(p_{o}\) loss as was case with friction
- As we lose \(p_{o}\) (~13% loss here)
  - less thrust
  - less work output

Choking

- Heating of a flow can lead to choking just like friction
- For supersonic flow, depending on back pressure, can get
  - underexpanded flow
  - overexpanded flow
  - shock inside duct