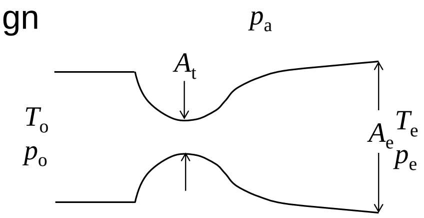
Rocket Propulsion Basics

Ideal Nozzles

Analysis Goal

- Determine performance (τ, I_{sp} or c_τ, c^*) of rocket nozzle based on
  - inflow properties
    - T_o, p_o
  - exit boundary conditions
    - p_a
  - nozzle geometry/design
    - $\varepsilon \equiv A_{exit}/A_{throat}$, A_{throat}

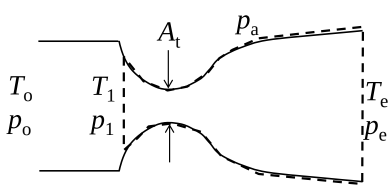


Idealizing Assumptions

1. working fluid is homogeneous gas (composition)
2. thermally and calorically perfect
3. adiabatic (negl.: wall heat transfer and radiation;) *few %*
4. reversible flow (negl.: boundary layers, viscous effects, shocks) ***some shocks, b.l.***
5. uniform properties in direction normal to flow ***depends on length***
6. quasi 1-d flow (only axial velocities)
7. steady flow ***depends on duration, combustion instabilities***

Analysis

- Consider nozzle control volume



- Mass conservation

$$\dot{m}_1 = \cancel{\frac{d}{dt} m_{CV}} + \dot{m}_e$$

assume steady flow

$$\dot{m}_1 = \dot{m}_e \equiv \dot{m}$$

- Energy Conservation

assume adiabatic

assume uniform flow

$$\dot{m}\left(h_1 + \frac{u_1^2}{2}\right) + \cancel{\dot{W}_{in}} = \cancel{\frac{d}{dt} E_{CV}} + \cancel{\dot{Q}_{out}} + \dot{m}\left(h_e + \frac{u_e^2}{2}\right)$$

$$\left(h_1 + \frac{u_1^2}{2}\right) = \left(h_e + \frac{u_e^2}{2}\right) \Rightarrow h_o = \text{constant}$$

stagnation enthalpy

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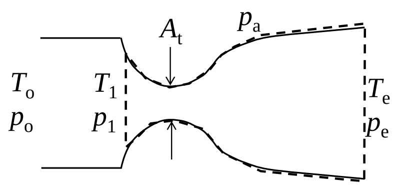
Exhaust Velocity

- So $u_e = \sqrt{2\left(h_o - h_e\right)}$

  tpg, cpg $= \sqrt{2c_p\left(T_o - T_e\right)}$

  "knowns" $\longrightarrow$ ✓ ✓ ?



$$T_o = T_e + u_e^2 / 2c_p$$

$$T_o \cong T_1$$ *velocity into nozzle typically low*

- How to find T_e?
  – use 2nd Law

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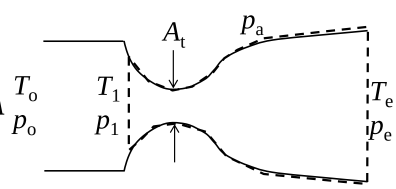
Temperature, Pressure Relations

- 2nd Law



rev. *steady* *adiabatic*

$$\dot{m}s_1 + \cancel{\dot{S}_{pr}} = \cancel{\frac{d}{dt}S_{CV}} + \dot{m}s_e + \int \cancel{\dot{q}''_{walls}}\, dA$$

$$\Rightarrow s_1 = s_e = s_o \quad \textit{isentropic}$$

Why assume rev.? *will produce $u_{e,max}$*

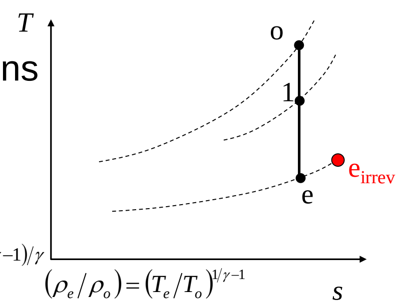
- Using TPG state relations

$$s_o - s_e = 0 = \int_e^o c_p \frac{dT}{T} - R\int_e^o \frac{dp}{p}$$

cpg $$= c_p \ln\frac{T_o}{T_e} - R\ln\frac{p_o}{p_e}$$

Isentropic Relations

$$\Rightarrow T_e/T_o = \left(p_e/p_o\right)^{R/c_p} = \left(p_e/p_o\right)^{(\gamma-1)/\gamma}$$



$$\left(\rho_e/\rho_o\right) = \left(T_e/T_o\right)^{1/\gamma-1}$$

$$\gamma \equiv c_p/c_v = c_p/\left(c_p - R\right)$$

Exhaust Velocity Optimization

- Combine results $u_e = \sqrt{2c_p\left(T_o - T_e\right)}$ $\qquad \dfrac{T_e}{T_o} = \left\{\dfrac{p_e}{p_o}\right\}^{(\gamma-1)/\gamma}$

$$u_e = \sqrt{2c_pT_o\left(1 - T_e/T_o\right)}$$

$$u_e = \sqrt{2c_pT_o\left(1 - \left\{\frac{p_e}{p_o}\right\}^{\frac{\gamma-1}{\gamma}}\right)}$$

- What can be done to increase u_e (and $\therefore I_{sp}$)?
  - higher T_o (~ $T_o^{1/2}$)

    limited by either energy available
    or material strength/survivability

Exhaust Velocity Optimization

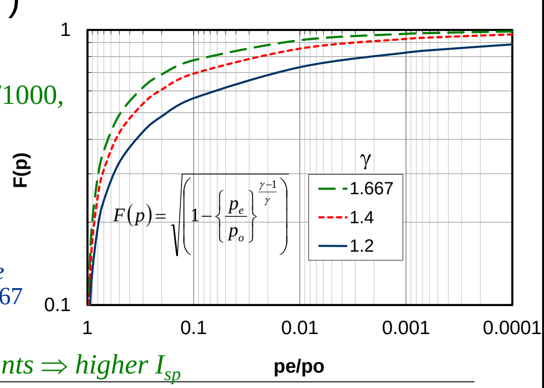
- What can be done to increase u_e (and $\therefore I_{sp}$)? $\qquad u_e = \sqrt{2c_pT_o\left(1 - \left\{\frac{p_e}{p_o}\right\}^{\frac{\gamma-1}{\gamma}}\right)}$
  - $\boxed{\text{higher } T_o}$ (~ $T_o^{1/2}$)
  - $\boxed{\text{higher } p_o / p_e}$

    for $p_e / p_o <$ 1/100 – 1/1000, diminishing returns
  - higher c_p

    $$c_p = \frac{\gamma}{\gamma - 1}\frac{\overline{R}}{MW}$$

    - $\boxed{\text{lower } \gamma}$ *range ~ 1.1-1.67*
    - $\boxed{\text{lower MW}}$

    light propellants $\Rightarrow$ higher I_{sp}



Thrust

- Already showed static thrust and equivalent force for accelerating case are same

$$\tau = \dot{m}u_e + \left(p_e - p_a\right)A_e$$

? ✓ ? ✓ ✓

need nozzle exit pressure and mass flowrate

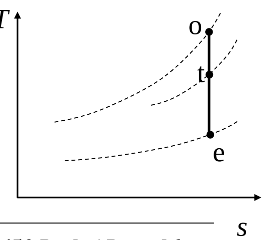
- Begin by reviewing pressure/temperature Mach relations (in isentropic nozzle)
  - adiabatic, no work $\Rightarrow h_o$ (or T_o) = const.
  - include rev. $\Rightarrow p_o$ = const.

$$T_o/T = 1 + u^2/2c_pT = 1 + \frac{\gamma-1}{2}M^2 \qquad \frac{u^2}{c_pT} = \frac{u(\gamma-1)}{\gamma RT}$$

$$\gamma RT = a^2$$

$$p_o/p = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1} \qquad M = u/a$$

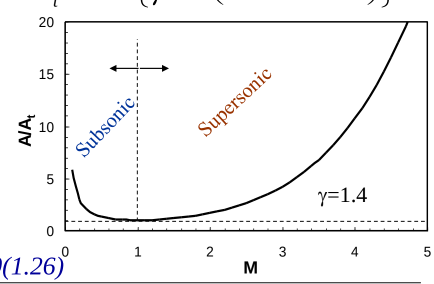
property variations only function of M



p_e from Nozzle Area Relations

- What determines M variation in nozzle?
  - cross-sectional area variation
- Constant mass flow rate through nozzle (steady) and combining isentropic (tpg/cpg) relations
  - nozzle flow starts subsonic
  - nozzle converges to accel to M = 1 (at throat)
  - nozzle diverges to accel to M > 1
  - large A_e/A_t ($\equiv \varepsilon$) to achieve high M

$$\frac{A}{A_t} = \frac{1}{M}\left\{\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right\}^{\gamma+1/2(\gamma-1)}$$



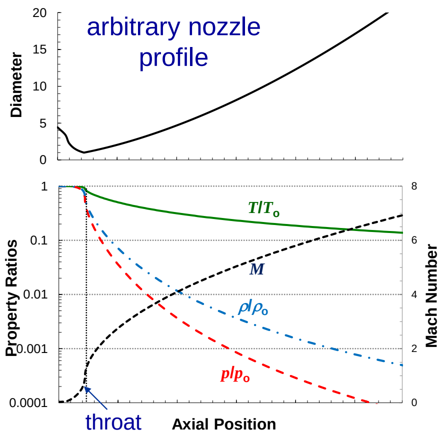
e.g., M=7⇒ε=100(γ=1.4) or =500(1.26)

typically $\varepsilon \leq 50{:}1$

p_e from Nozzle Property Variations

- How do flow properties vary through nozzle?
  - example for γ=1.26
- All static properties drop along nozzle
  - typically rapid near throat
  - least ? ***T***
  - most ? ***p***
- $p_e = fn(p_o, M_e) = fn(p_o, \varepsilon)$

  $\Rightarrow$only one ε will produce $p_e = p_a$ (perfectly expanded) for given p_o



Mass Flowrate

$$\tau = \dot{m} u_e + (p_e - p_a) A_e$$

? ✓ ? ✓ ✓

- Now that we found exit pressure, need mass flowrate to get thrust
- From continuity $\dot{m} = \rho\ u\ A$

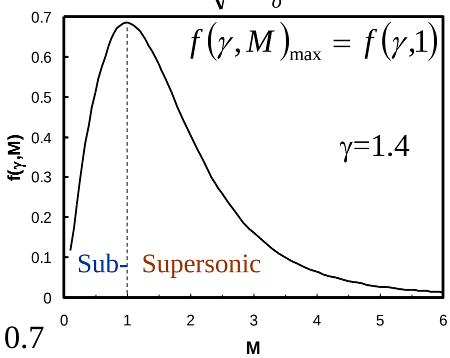
  For tpg $\quad = \dfrac{p}{RT}\left(M\sqrt{\gamma RT}\right)A$
- Using (tpg + cpg) isentropic relations

$$\dot{m}/A = \frac{p_o}{\sqrt{RT_o}} M \sqrt{\frac{\gamma}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma+1/\gamma-1}}} = \frac{p_o}{\sqrt{RT_o}}\ f(\gamma, M)$$

Choked Nozzle

$$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}} f(\gamma, M)$$

- For fixed p_o and T_o mass flux at given nozzle cross-section controlled by $f(\gamma, M)$
  - highest mass flux at throat (M will be closest to 1 there)
  - for given A_t, maximum flowrate when nozzle is choked ($M_t = 1$)



$$f(\gamma,1) = \begin{cases} 0.726 & 1.67 \\ 0.685 & 1.4 \\ 0.660 & 1.26 \end{cases} \approx 0.7 \quad \text{for } \gamma$$

Choked Mass Flow Rate

- So we have
$$\dot{m}_{choked} \propto A_t \frac{p_o}{\sqrt{RT_o}}$$
  - can increase with
    - higher p_o (linearly)
    - lower T_o ($T_o^{½}$)
    - larger throat (& nozzle) (linear with A_t)
    - heavier molecular weight ($R^{½} \propto 1/MW^{½}$)
- What does it take to choke the nozzle?
  - will show later requires $p_o/p_a >$ critical value that is function of ε
  - for rocket nozzles, critical value not much bigger than 1

Thrust Optimization (Ideal)

- So for perfectly expanded, ideal nozzle

$$\tau = \dot{m} u_e$$

$$\dot{m} \propto A_t \frac{p_o}{\sqrt{RT_o}} f(\gamma) \qquad u_e = \sqrt{2c_p T_o \left(1 - \left\{\frac{p_e}{p_o}\right\}^{\frac{\gamma-1}{\gamma}}\right)}$$

- What will increase thrust?
  - A_t ?
  - p_o ? $\quad \tau \propto p_o \sqrt{1 - \{p_e/p_o\}^{\frac{\gamma-1}{\gamma}}}$
  - T_o ? $\quad \tau \propto \sqrt{T_o/T_o} \neq fn(T_o)$
  - MW ? $\quad \tau \propto \sqrt{c_p/R} = \sqrt{\gamma/(\gamma-1)}$
  - γ ? $\quad \tau \propto \sqrt{\frac{\gamma}{\gamma-1}\gamma\left(\frac{\gamma+1}{2}\right)^{\gamma+1/1-\gamma}}$

neglecting p term