Control Volume Derivation

• How to convert our relationships for a closed system (control mass) to an open system (control volume)

• For mass conservation, our control mass “law” was

\[
\frac{dm_{sys}}{dt} = 0
\]

• In integral form (integrating over control mass),

\[
\frac{d}{dt} \int_{CM} \rho dV = 0
\]
Control Mass/Control Volume

- Consider general control mass and control volumes that are moving in time; coincide at time \( t \)

- Want to see how to convert CM law to CV law

\[
\frac{d}{dt} \int_{CM} \rho(\vec{x}, t) dV = 0
\]

- Must integrate 3-d integral over time dependent domain

\[ v_{rel} = \text{relative velocity of material crossing CV's CS} \]

\[ dA = \text{differential area element on CV's CS} \]
Time Derivative of 3-d Integral

- Start with standard limit value definition of derivative

\[
\frac{d}{dt} \int_{CM(t)} \rho(\bar{x}, t) dV = \lim_{\Delta t \to 0} \left[ \frac{\int_{CM(t+\Delta t)} \rho(\bar{x}, t + \Delta t) dV - \int_{CM(t)} \rho(\bar{x}, t) dV}{\Delta t} \right]
\]

but

\[
\int_{CM(t+\Delta t)} \rho(\bar{x}, t + \Delta t) dV = \int_{CV(t+\Delta t)} \rho(\bar{x}, t + \Delta t) dV + \int_{\text{Shaded Region}} \rho(\bar{x}, t + \Delta t) dV
\]

\[
\frac{d}{dt} \int_{CM(t)} \rho(\bar{x}, t) dV = \lim_{\Delta t \to 0} \left[ \frac{\int_{CV(t+\Delta t)} \rho(\bar{x}, t + \Delta t) dV - \int_{CV(t) = \text{CM}(t)} \rho(\bar{x}, t) dV}{\Delta t} \right] + \frac{1}{\Delta t} \int_{\text{Shaded Region}} \rho(\bar{x}, t + \Delta t) dV
\]
Time Derivative of 3-d Integral (con’t)

- But \[ \] term is definition of derivative
  \[
  \lim_{\Delta t \to 0} \left[ \frac{\int \rho(\bar{x}, t + \Delta t) dV - \int \rho(\bar{x}, t) dV}{\Delta t} \right] = \frac{d}{dt} \int_{CV(t)} \rho dV
  \]

- Use \( dV = \vec{v}_{rel} \Delta t \cdot \vec{n} dA \) with \( \vec{n} \) a vector normal to \( dA \) and pointed outward

- Shaded region term becomes

\[
\lim_{\Delta t \to 0} \left[ \frac{1}{\Delta t} \int_{\text{Shaded Region}} \rho(\bar{x}, t + \Delta t) dV \right] = \lim_{\Delta t \to 0} \left[ \frac{\Delta t}{\Delta t} \int_{\text{CS of CV at } t} \rho(\vec{v}_{rel} \cdot \vec{n}) dA \right] = \int_{\text{CS}(t)} \rho(\vec{v}_{rel} \cdot \vec{n}) dA
\]
Control Volume Form of Mass Conservation

- From previous two equations, we have

\[
\frac{d}{dt} \int_{CM(t)} \rho dV = \frac{d}{dt} \int_{CV(t)} \rho dV + \int_{CS(t)} \rho (v_{rel} \cdot \bar{n}) dA
\]

- Applying mass conservation (LHS=0)

\[
0 = \frac{d}{dt} \int_{CV(t)} \rho dV + \int_{CS(t)} \rho (v_{rel} \cdot \bar{n}) dA
\]

Production rate of mass, \( \dot{P}_{mass} \)

Time rate of change of mass inside CV

Net outward mass flow rate crossing CS

Don’t have to know mass distribution in CV

Production rate of mass, \( \dot{m}_{CV} \)

\[
\dot{P}_{mass} = 0 = \frac{dm_{CV}}{dt} + \sum_{\text{outlets}} m - \sum_{\text{inlets}} m
\]
Simplifications

- **Uniform flow (at CS)**

\[
\int_{CS(t)} \rho (\vec{v}_{rel} \cdot \vec{n}) dA = \sum_{outlets} \rho v_{rel}A - \sum_{inlets} \rho v_{rel}A
\]

i.e., \( \dot{m} = \rho v_{rel}A \)

+ Working in frame of reference where CS not moving

\[
\Rightarrow \sum_{inlets} \rho vA - \sum_{outlets} \rho vA = 0
\]

- **Steady-State**

\[
\frac{d}{dt} \int_{CV(t)} \rho dV = 0 \Rightarrow \sum_m = \sum_m
\]
Simplifications (con’t)

- Transient, integrate over fixed time

\[
\int_{t_1}^{t_2} \left[ \frac{d}{dt} \int_{CV(t)} \rho dV \right] dt = \int_{t_1}^{t_2} \left[ \sum_{\text{inlets}} \dot{m} - \sum_{\text{outlets}} \dot{m} \right] dt
\]

\[
\int_{CV(t_2)} \rho dV - \int_{CV(t_1)} \rho dV = \int_{t_1}^{t_2} \left[ \sum_{\text{inlets}} \frac{dm}{dt} - \sum_{\text{outlets}} \frac{dm}{dt} \right] dt
\]

\[
m_{CV,2} - m_{CV,1} = \sum_{\text{inlets}} \Delta m_{12} - \sum_{\text{outlets}} \Delta m_{12}
\]

Change of mass in CV between \( t_1 \) and \( t_2 \)  
Net amount of mass entering CV between \( t_1 \) and \( t_2 \)
In-Class Problems

1. Nitrogen at 200 kPa and 250°C flows through a 35 mm diam. pipe at 20 m/s. Find the mass flow rate of nitrogen through the pipe.

2. Liquid water enters the square duct shown with an average velocity 10 m/s. Determine the average velocity and mass flow rate at the exit.

3. Air flows in a circular pipe with a velocity of 20 m/s. Around the pipe, in an annulus, is a 2nd flow of air, with a velocity of 40 m/s. Both flows exhaust into a 15 cm diam. pipe. If the flow at e is uniform, determine the flow velocity at e. Assume the air density is constant.
Differential Form of Mass Conservation
For Quasi-1D, Steady Flow

- Assume flow velocity is 1-D (only variation in x) in non-constant area, differential volume
  \[ \dot{m} = \text{constant} = \rho v A \]
  \[ 0 = d(\rho v A) \]
  \[ 0 = v A d\rho + \rho A dv + \rho v dA \]

\[
0 = \frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = \frac{d\rho}{\rho} + \frac{1}{2} \frac{d(v^2)}{A} + \frac{dA}{A}
\]

- Compare to continuity

\[
0 = \frac{d\rho}{dt} + \left[ \frac{d(\rho v_x)}{dx} + \frac{d(\rho v_y)}{dy} + \frac{d(\rho v_z)}{dz} \right] = \frac{d(\rho v)}{dx} \Rightarrow \left( \text{strictly, } \frac{dA}{dx} = 0 \right)
Reynolds Transport Theorem

- Provides general form for converting from CM to CV conservation laws
- For given extensive property B, with intensive version $\beta$ (something per mass), that follows a “conservation” law can show

$$\left. \frac{dB}{dt} \right|_{CM} = \frac{d}{dt} \int_{CV} \rho \beta dV + \int_{CS} \rho \beta (\vec{v}_{rel} \cdot \vec{n}) dA$$

Replace with appropriate Control Mass Conservation Law

- Will also lead to a **PICO** relationship
  
  Production + Input = Change (in time) + Output
Reynolds Transport Theorem: Example

- Example, **linear momentum**
  \[ \vec{B} = m\vec{v}, \quad \vec{\beta} = \vec{v} \]

- **RTT** then gives
  \[
  \frac{d(m\vec{v})}{dt} \bigg|_{CM} = \frac{d}{dt} \int \rho \vec{v} dV + \int \rho \vec{v}(\vec{v}_{rel} \cdot \vec{n}) dA
  \]

- **Use Newton’s Law**
  - e.g., gravity
  - e.g., pressure (\(\int p dA\)), shear stress

  \[
  \frac{d(m\vec{v})}{dt} \bigg|_{CM} = \sum \vec{F}_{on\ CV} = \sum \vec{F}_{body\ on\ CV} + \sum \vec{F}_{surface\ on\ CS}\left(= \dot{P}_{momentum}\right)
  \]

  \[
  \dot{P}_{momentum} = \sum \vec{F}_{on\ CV} = \frac{d}{dt} \int \rho \vec{v} dV + \int \rho \vec{v}(\vec{v}_{rel} \cdot \vec{n}) dA
  \]
Momentum Conservation: 1-D Flow

- For steady inviscid flow, no body forces

\[
\sum_{0}^{\text{body on CV}} \vec{F}_x + \sum_{\text{solid CS}} \vec{F}_x \text{stresses from solid CS} - \int_{\text{CS}} p \cdot \vec{n} \, dA = 0
\]

\[
\frac{d}{dt} \int_{\text{CV}} \rho \vec{v} \, dV + \int_{\text{CS}} \rho \vec{v} (\vec{v}_{\text{rel}} \cdot \vec{n}) \, dA
\]

\[
\vec{F}_{\text{x shear stress on CS}} - \int_{\text{CS}} p \cdot \vec{n} \, dA = \sum_{\text{outlets}} \dot{m} \vec{v} - \sum_{\text{inlets}} \dot{m} \vec{v}
\]

\[
\vec{F}_x + p_1 A_1 - p_2 A_2 + p_{\text{amb}} (A_2 - A_1) = \dot{m} v_2 - \dot{m} v_1 = \rho_2 v_2^2 A_2 - \rho_1 v_1^2 A_1
\]

\[
\vec{F}_x + (p_1 - p_{\text{amb}}) A_1 - (p_2 - p_{\text{amb}}) A_2 = \dot{m} v_2 - \dot{m} v_1 = \rho_2 v_2^2 A_2 - \rho_1 v_1^2 A_1
\]
Momentum Conservation: 1-D Flow

• Differential form

\[-\tau_x L_p \, dx + pA - (p + dp)(A - dA)\]

\[-\left( p + \frac{dp}{2} \right) dA = \dot{m}(v + dv) - \dot{m}v\]

\[-\tau_x L_p \, dx - Adp = \dot{m}dv = \rho v A dv\]

\[\tau_x \frac{L_p}{p} \, dx + \frac{dp}{A} + \frac{\rho v^2}{2p} \frac{dv^2}{v^2} = 0\]

- For steady, no body forces and shear stress defined to be in -x direction
Flow Rates and Fluxes

• **Flow rate of property** that is **carried by the mass** crossing the control surface (in some direction) is

\[ B = \int \rho \beta (\vec{v}_{\text{rel}} \cdot \vec{n}) dA \]

\[ \text{e.g., } \dot{m} = \rho v A \text{ (kg/s)} , \]

\[ \dot{m} v = \rho v^2 A \text{ (N) for 1-D flow with stationary CV} \]

• The **flux** of that same property is given by

\[ \dot{B}'' = \rho \beta (\vec{v}_{\text{rel}} \cdot \vec{n}) \]

\[ \text{e.g., } \rho v \left( \text{mass flux, kg/s/m}^2 \right) , \]

\[ \rho v^2 \left( \text{momentum flux, N/m}^2 \right) \text{ for nonmoving CS} \]