# Review Problems:

## Ideal Gas Dynamics, Kinetic Theory, Statistical Mechanics

- Homework solutions should be neat and logically presented, see format requirements ([seitzman.gatech.edu/classes/ae6050/homeworkformat.html](http://seitzman.gatech.edu/classes/ae6050/homeworkformat.html)). **Please note the requirement to draw some IMPLICATIONS/CONCLUSIONS from the results of each problem.** These could be how your findings would impact design/operation of a practical device, a comparison of the results of different parts of a problem, a physical interpretation of the results, etc.

- To receive credit, show ALL work in the format described above. If you use equations from the notes, the class textbook or another book, please cite the reference.

## 1. Isentropic Nozzle

a) Write a complete set of expressions (DO NOT derive them, just cite your source) for a thermally and calorically perfect, nonreacting gas as it isentropically expands through a supersonic nozzle that relate: the temperature \( T \), pressure \( p \), density \( \rho \), Mach number \( M \) of a flow; the local cross-sectional area \( A \) and throat area \( A_t \) of the nozzle; and any necessary upstream **reservoir** conditions. Make sure to include any additional assumptions that these expressions use (i.e., assumptions that are not specifically given to you in this problem).

b) Plot (not by hand!) \( T \), \( p \), \( \rho \) and \( M \) vs. \( A/A_t \) for a mixture that is 50% H\(_2\) and 50% Xe by mole through such a nozzle as it expands from a reservoir \( (M\sim0) \) to an exit Mach number of at least 7 with reservoir conditions of 1300 K and 25 atm. Base your specific heat values for this mixture only on the modes you expect to be near fully-excited (and therefore approximately constant) over the range of temperatures covered.

## 2. Normal Shocks

a) Write expressions (again no derivations, just cite your sources) that give the temperature ratio, \( T_2/T_1 \), pressure ratio, \( p_2/p_1 \), density ratio \( \rho_2/\rho_1 \), and normalized entropy increase, \( (s_2 − s_1)/R \), for a normal shock in a thermally and calorically perfect, nonreacting gas as a function of the shock Mach number, \( M_1 \). Again, be sure to include any additional assumptions used.

b) Plot the 1) temperature ratio, 2) pressure ratio, 3) density ratio, 4) normalized entropy increase and 5) velocity ratio for normal shocks with Mach numbers up to at least 15 using the gas mixture described in Problem 1, with initial conditions of 450K and 1 atm. Base your specific heat values for this mixture only on the modes you expect to be near fully-excited at the pre-shock conditions.
3. Collision Frequency and Mean Free Path

The mean free path ($\lambda$) is the average distance a molecule travels between successive collisions. It can be approximated as $\lambda = \overline{C}/\Theta$, the average molecular speed divided by the collision frequency.

For this problem, let’s say a single oxygen ($O_2$) molecule experiences elastic collisions (i.e., collisions where only momentum or translational energy is exchanged) with all other molecules in a pure $O_2$ gas with a frequency of $5 \times 10^9$ sec$^{-1}$ at “normal” temperature and pressure conditions (NTP = 20°C and 1 atm).

a) Estimate the mean free path of an $O_2$ molecule at NTP.

b) Estimate the elastic collision cross-section (in units of cm$^2$) for $O_2$ based on the values given in this problem.

c) Estimate the elastic collision frequency for a pure $O_2$ gas if we kept the density constant but increased the pressure by a factor of 4?

d) Estimate the mean free path for $O_2$ for the same conditions used in part c).

4. Bimolecular Collision Rate

Derive an expression for the fraction of bimolecular collisions (during a fixed time interval) that have a relative kinetic energy of the colliding molecules that is greater than $n$ times the average relative kinetic energy (given by $\frac{1}{2} kT$). Then plot this fraction as a function of $n$. You may assume a constant collision cross-section.

5. Equilibrium Enthalpy

Consider a diatomic gas that can be modeled as a rigid rotor (with $\theta_r=3.5$ K) and harmonic oscillator (with $\theta_v=2700$K), and with four electronic energy levels ($g_0=1$; $g_1=3 \theta_1=600$ K; $g_2=3$, $\theta_2=2700$ K; $g_3=5$, $\theta_3=100,000$ K).

a) Write a simplified expression for the normalized enthalpy ($h/R$) of this gas as a function of temperature that would be valid for temperatures above 200 K. Your expression should include any necessary molecular properties given above BUT do not enter numbers for the properties, use variables/symbols.

b) Using the standard thermodynamic definition of $c_p$ for a thermally perfect gas, i.e., $c_p=dh/dT$, plot the specific heat for this gas from 200 K to 4500 K based on a numerical differentiation of your enthalpy (i.e., do not analytically differentiate your result from part a, but rather calculate values for $h/R$ at different temperatures and use a differencing scheme to calculate its derivative).
6. **Specific Heat and Fully-Excited Conditions**

The most general form of the partition function for a perfect gas is

\[ Q = g_0 + \sum_{i=1}^{n} g_i e^{-\theta_i/T} \]

where the lowest energy level (denoted 0) has zero energy, and \( \theta_i \) is the characteristic temperature of the \( i \)th energy level (\( = \varepsilon_i/k \)).

a) Using the above partition function expression WITHOUT any simplifying assumptions, derive an expression for the normalized specific heat \( (c_v/R) \) of a perfect gas. Your answer will contain various series summations. **SUGGESTION:** while you could start with the expression given in class for \( E \) as a function of \( \partial(\ln Q)/\partial T \), it might be easier to start with simpler expression:

\[ E = \sum N_i \varepsilon_i = N \sum f_i \varepsilon_i \]

b) Use the result from (a) to plot \( c_v,rot/R \) versus \( T/\theta_r \) for a linear rigid rotor starting from \(~0K\) and extending at least until the rotational energy mode is “fully-excited”. Assume \( J=0 \) is the ground rotational energy level. **Also** plot the value of \( Q \) as a function of \( T/\theta_r \). (NOTE: you CAN NOT use the assumption that \( Q=T/\theta_r \).)

c) At what temperatures would you say that the rotational mode is fully excited for the following gases: 1) \( H_2 \), 2) OH and 3) \( O_2 \).