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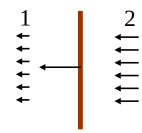
Normal Shock Waves



- For a **normal shock**
  - wave is perpendicular to flow (propagation) direction
- Shock is nonequilibrium process internally, but assume
  - flow *before shock* (1) is in equilibrium
  - flow *after shock* (2) is in equilibrium

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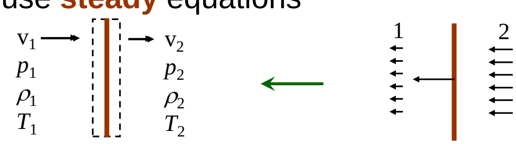
Approach to Finding Shock Properties

- Start with stationary shock (or solve problem in reference frame moving with shock)
  - $\Rightarrow$ can use **steady** equations



- Use **control volume** analysis
  - only need to consider properties before and after shock (equilibrium)
- Equations first studied by Rankine (~1870) and Hugoniot (~1877)

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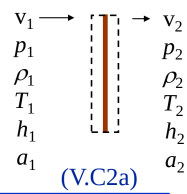
Governing Equations



- Conservation and state equations
  - 1d, steady, inviscid except inside shock, adiabatic, only flow work

Mass $$\frac{\dot{m}}{A} = \rho_1 v_1 = \rho_2 v_2 \quad \text{(V.C1)}$$

Momentum $$p_1 A - p_2 A = \dot{m}_2 v_2 - \dot{m}_1 v_1 \longrightarrow p_1 - p_2 = \frac{\dot{m}}{A}(v_2 - v_1) \quad \text{(V.C2a)}$$

$$p_1 A - p_2 A = \rho_2 v_2^2 A - \rho_1 v_1^2 A \longrightarrow p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \quad \text{(V.C2b)}$$

Energy $$h_1 + v_1^2/2 = h_2 + v_2^2/2 = h_o \quad \text{(V.C3)}$$

Perfect Gas State Eqns. $$p = \rho RT \text{ (tpg)} \qquad dh = c_p dT \text{ from III.6} \qquad a^2 = \gamma RT \text{ from V.A3}$$

- **6 equations, 6 unknowns**

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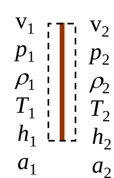
Shock Density Ratio



- Combine energy, momentum, mass

(V.C3) $\longrightarrow$ $$h_2 - h_1 = \frac{1}{2}\left(v_1^2 - v_2^2\right) = \frac{1}{2}\left(v_1 - v_2\right)\left(v_1 + v_2\right)$$

(V.C2a) $$\frac{p_1 - p_2}{\dot{m}/A} = (v_2 - v_1)$$

$$\frac{\dot{m}}{A}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = v_1 + v_2 \quad \text{(V.C1)}$$

Shock Hugoniot Eq.

(V.C4) $$h_2 - h_1 = \frac{1}{2}\left(p_2 - p_1\right)\left(1/\rho_1 + 1/\rho_2\right)$$

true for all simple comp. substances

tpg/cpg (III.6) (tpg)

$$h_2 - h_1 = c_p(T_2 - T_1) = \frac{R\gamma}{\gamma - 1}\left(p_2/R\rho_2 - p_1/R\rho_1\right)$$

only functions of p and ρ

- Solve for density ratio $\longrightarrow$

$$\frac{\rho_2}{\rho_1} = \frac{1 + \dfrac{\gamma+1}{\gamma-1}\dfrac{p_2}{p_1}}{\dfrac{\gamma+1}{\gamma-1} + \dfrac{p_2}{p_1}} \quad \text{(V.C5)}$$

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Entropy Change Across Shock

- For tpg/cpg, entropy state equation

p_1, ρ_1, s_1 | p_2, ρ_2, s_2

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} - R\ln\frac{\rho_2}{\rho_1}$$

$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{T_2}{T_1}\left(\frac{\rho_2}{\rho_1}\right)^{-(\gamma-1)}\right] = \ln\left[\frac{p_2/\rho_2}{p_1/\rho_1}\left(\frac{\rho_2}{\rho_1}\right)^{-(\gamma-1)}\right]$$

(tpg)

$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right] \quad \text{(V.C6)}$$

entropy change as function of pressure and density ratios across shock

Entropy Change Across Shock

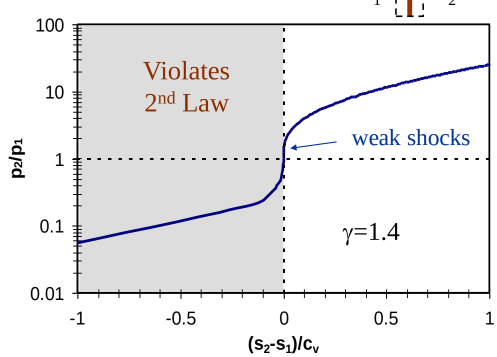
$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right] \qquad \frac{\rho_2}{\rho_1} = \frac{1 + \frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}$$

p_1, ρ_1, s_1 | p_2, ρ_2, s_2

- Combine s_2/s_1 result with ρ_2/ρ_1 (from V.C5)

solution with $p_2<p_1$ violates 2nd Law

⇒no expansion "shocks"



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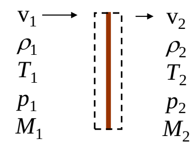
Mach Number Relations: v, ρ



- General problem is to find change in all properties across shock
- Start by finding changes based on M_1, M_2
  - then find M_2 as function of M_1
- **Velocity ratio,** v_2/v_1

$$\frac{v_2}{v_1} = \frac{M_2 a_2}{M_1 a_1} \longrightarrow \boxed{\frac{v_2}{v_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}} \qquad \text{(V.C7)}$$

- **Density ratio,** ρ_2/ρ_1

$$\text{mass} \qquad \rho_1 v_1 = \rho_2 v_2 \longrightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}} \qquad \text{(V.C8)}$$

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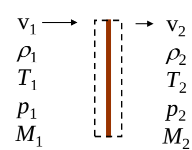
Mach Number Relations: T



- **Temperature ratio,** T_2/T_1

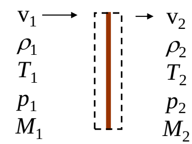
$$\text{energy} \qquad h_{o1} = h_{o2} \Rightarrow T_{o1} = T_{o2}$$

$$\frac{T_2}{T_1} = \frac{T_2/T_{o2}}{T_1/T_{o1}} \frac{T_{o2}}{T_{o1}} \quad (\text{with } \tfrac{T_{o2}}{T_{o1}} \to 1)$$

$$= \frac{1\Big/\left(1+\frac{\gamma-1}{2}M_2^2\right)}{1\Big/\left(1+\frac{\gamma-1}{2}M_1^2\right)} \longrightarrow \boxed{\frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)}{\left(1+\frac{\gamma-1}{2}M_2^2\right)}} \qquad \text{(V.C9)}$$

Mach Number Relations: p

- **Pressure ratio**, p_2/p_1



momentum (V.C2b)

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2$$

$$\rho v^2 = \rho\left(M^2 a^2\right) = \rho\left(M^2 \gamma R T\right) = \rho\left(M^2 \gamma \frac{p}{\rho}\right) = p\gamma M^2$$

$$p_1 + p_1 \gamma M_1^2 = p_2 + p_2 \gamma M_2^2$$

$$p_1\left(1+\gamma M_1^2\right) = p_2\left(1+\gamma M_2^2\right)$$

$$\frac{p_2}{p_1} = \frac{\left(1+\gamma M_1^2\right)}{\left(1+\gamma M_2^2\right)} \quad \text{(V.C10)}$$

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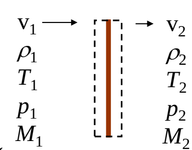
Mach Number Relations: M

- **Mach Number**, $M_2 = f(M_1)$
  - combine all eqn's.



mass (V.C1)

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1}$$

tpg

$$\frac{\rho_2}{\rho_1} = \frac{p_2/RT_2}{p_1/RT_1}$$

M and a (V.A3)

$$\frac{M_1\sqrt{\gamma R T_1}}{M_2\sqrt{\gamma R T_2}} = \frac{v_1}{v_2}$$

mom. (V.C9)

$$\frac{\left(1+\gamma M_2^2\right)}{\left(1+\gamma M_1^2\right)} = \frac{p_2}{p_1}$$

energy (V.C10)

$$\frac{T_2}{T_1} = \left(1+\frac{\gamma-1}{2}M_1^2\right)\Big/\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

$$\frac{M_2}{1+\gamma M_2^2}\sqrt{1+\frac{\gamma-1}{2}M_2^2} = \frac{M_1}{1+\gamma M_1^2}\sqrt{1+\frac{\gamma-1}{2}M_1^2}$$

expression for M_2 as function of M_1 - can ***solve***

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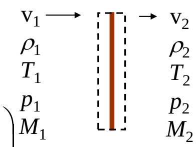
Mach Number Relations: M (con't)

- Remove square roots by squaring both sides, solve resulting quadratic



$$\frac{M_2^2}{\left(1+\gamma M_2^2\right)^2}\left(1+\frac{\gamma-1}{2}M_2^2\right)=\underbrace{\frac{M_1^2}{\left(1+\gamma M_1^2\right)^2}\left(1+\frac{\gamma-1}{2}M_1^2\right)}_{\equiv g(M_1)}$$

$$M_2^4\left[\frac{\gamma-1}{2}-\gamma^2 g(M_1)\right]+M_2^2\left[1-2\gamma g(M_1)\right]-g(M_1)=0$$

$$M_2^2=\frac{M_1^2+\frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2-1} \quad \text{or} \quad M_1^2 \quad \text{(V.C11)}$$

"No shock" solution ($M_2^2 = M_1^2$)

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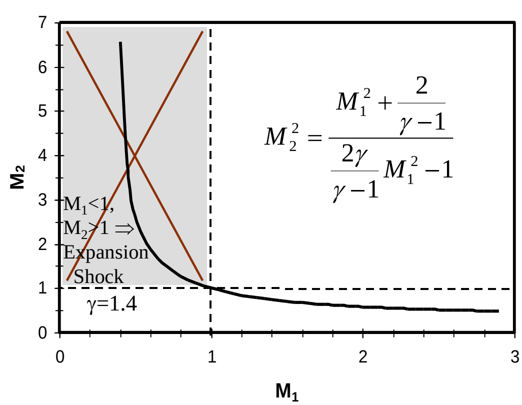
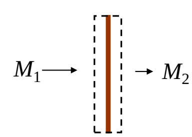
Mach Change Across Normal Shock





*in shock's reference frame, flow **after** normal shock is always **subsonic***

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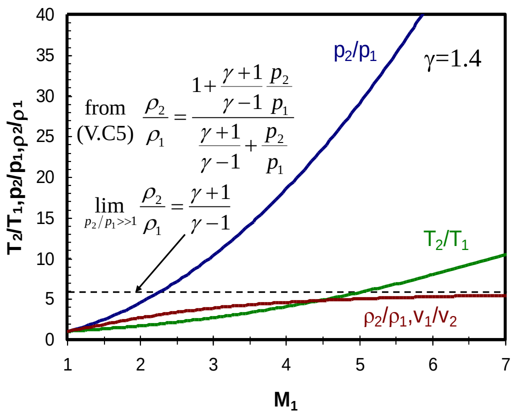
Property Ratios - Results



$$\frac{\rho_2}{\rho_1}=\frac{1+\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}} \quad \text{from (V.C5)}$$

$$\lim_{p_2/p_1>>1}\frac{\rho_2}{\rho_1}=\frac{\gamma+1}{\gamma-1}$$

$v_1 \rightarrow$, ρ_1, T_1, p_1, M_1 | shock | $\rightarrow v_2$, ρ_2, T_2, p_2, M_2

- T, p and ρ **increase** across shock and ratios increase with M_1
- v **decreases**
- p ratio increase across normal shock is **greatest static property change**
- ρ and v ratios approach **limit**

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Stagnation Properties Across Shock

- **Stagnation Temperature**, $T_{o2}=T_{o1}$
- **Stagnation Pressure**

$M_1 \rightarrow$, T_{o1}, p_{o1}, ρ_{o1} | shock | $\rightarrow M_2$, T_{o2}, p_{o2}, ρ_{o2}

$$\frac{p_{o2}}{p_{o1}}=\frac{p_{o2}/p_2}{p_{o1}/p_1}\frac{p_2}{p_1}=\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}$$

from (V.C10)

$$\frac{p_2}{p_1}=\frac{\left(1+\gamma M_1^2\right)}{\left(1+\gamma M_2^2\right)}$$

M_2 from (V.C11)

$$\frac{p_2}{p_1}=\frac{2\gamma}{\gamma+1}M_1^2-\frac{\gamma-1}{\gamma+1} \qquad \text{(V.C12)}$$

$$\frac{p_{o2}}{p_{o1}}=\left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2-\frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}} \qquad \text{(V.C13)}$$

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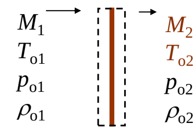
Stagnation Pressure

- **Other Useful Expressions**



$$\frac{p_{o2}}{p_{o1}} = \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2} \right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1}$$

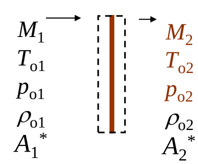
$$\boxed{\frac{p_{o2}}{p_{o1}} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1}} \quad \text{(V.C14)}$$

$$\frac{p_{o2}}{p_1} = \frac{p_{o2}}{p_2}\frac{p_2}{p_1}$$

$$\boxed{\frac{p_{o2}}{p_1} = \left(1+\frac{\gamma-1}{2}M_2^2\right)^{\frac{\gamma}{\gamma-1}} \frac{p_2}{p_1}} \quad \text{(V.C15)}$$

Stagnation Properties (con't)

- **Stagnation Density,** ρ_{o2}/ρ_{o1}



tpg $$\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}} \bigg/ \frac{RT_{o2}}{RT_{o1}} \quad (1)$$

$$\boxed{\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}}} \quad \text{(V.C16)}$$

- **Sonic Area Ratio,** A_2^*/A_1^*

from (V.B4) $$\frac{\dot{m}_2}{\dot{m}_1} = \frac{A_2^*}{A_1^*}\,\frac{p_{o2}/\sqrt{RT_{o2}}}{p_{o1}/\sqrt{RT_{o1}}}\,\frac{f(\gamma)}{f(\gamma)} \quad (1,\ 1,\ 1)$$

$$\Rightarrow \boxed{\frac{A_2^*}{A_1^*} = \frac{p_{o1}}{p_{o2}}} \quad \text{(V.C17)}$$

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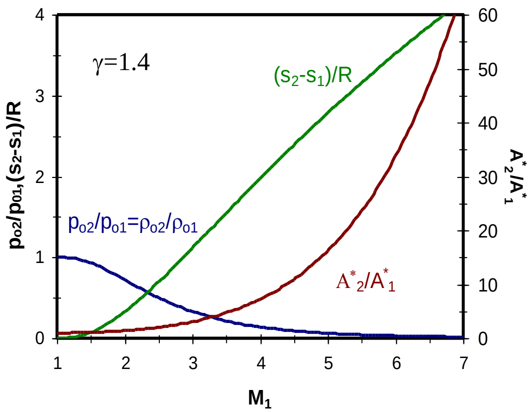
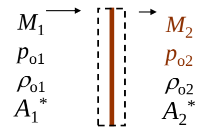
Stag. Press., Entropy and A* - Results





- *Stagnation pressure and stagnation density* drop
- Entropy increases
- Sonic area increases
  - larger throat required after shock to reach sonic flow (same mass flowrate, lower p_o)

$\dot{m} \propto A^* p_o$

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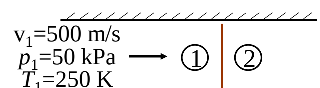
Examples: Problem 1



- **Given**: Air flowing through constant area duct encounters stationary shock
  - oncoming air 500 m/s, 50 kPa, 250K
- **Find**:
  - M_2, T_2, p_2, v_2
  - p_{o2}, T_{o2} (relative to shock/duct)
- **Assume**: air is tpg/cpg with γ=1.4

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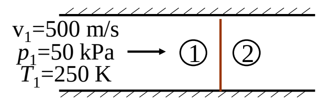
Solution: Problem 1

- **Analysis:**
  - first calculate M_1



$$M_1 = \frac{v_1}{\sqrt{\gamma R T}} = \frac{500\, m/s}{20\sqrt{250}\, m/s} = 1.58$$

  - can use equations or tables (e.g., App. B **for γ=1.4**)

M_1	M_2	p_2/p_1	T_2/T_1	p_{o2}/p_{o1}	ρ_2/ρ_1	A^*_2/A^*_1	p_{o2}/p_1
1.50	0.7011	2.458	1.320	0.9298	1.862	1.076	3.413
1.51	0.6976	2.493	1.327	0.9266	1.879	1.079	3.451
1.52	0.6941	2.529	1.334	0.9233	1.896	1.083	3.489
1.53	0.6907	2.564	1.340	0.9200	1.913	1.087	3.528
1.54	0.6874	2.600	1.347	0.9166	1.930	1.091	3.567
1.55	0.6841	2.636	1.354	0.9132	1.947	1.095	3.606
1.56	0.6809	2.673	1.361	0.9097	1.964	1.099	3.645
1.57	0.6777	2.709	1.367	0.9062	1.981	1.104	3.685
1.58	0.6746	2.746	1.374	0.9026	1.998	1.108	3.724
1.59	0.6715	2.783	1.381	0.8989	2.015	1.112	3.765

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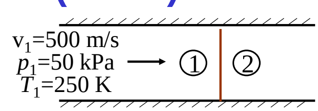
Solution: Problem 1 (con't.)

- **Analysis (con't):**

$M_1 = 1.58$



- M_2:V.C11 $\quad M_2^2 = \left(1.58^2 + \frac{2}{1.4-1}\right) \Big/ \left(\frac{2(1.4)}{1.4-1}1.58^2 - 1\right)$ **M_2=0.675**
- T_2: V.C9 $\quad \frac{T_2}{T_1} = \left(1 + \frac{1.4-1}{2}1.58^2\right) \Big/ \left(1 + \frac{1.4-1}{2}0.675^2\right) = 1.374$ **T_2=344 K**
- p_2: V.C12 $\quad \frac{p_2}{p_1} = \left[2\frac{1.4}{1.4-1}\right]1.58^2 - \left[\frac{1.4-1}{1.4+1}\right] = 2.746$ **p_2=137 kPa**
- v_2: V.C7 $\quad \frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = (1.58/0.675)\sqrt{1/1.374} = 1.998$ **v_2=250 m/s**
- p_{o2}:V.C15 $\quad \frac{p_{o2}}{p_1} = \left(1 + \frac{1.4-1}{2}0.675^2\right)^{3.5} 2.746 = 3.724$ **p_{o2}=186 kPa**
  V.A6 **p_{o1}=206 kPa**
- T_{o2}: V.A5 $\quad T_{o2} = T_{o1} = 250K\left(1 + \frac{1.4-1}{2}1.58^2\right) = (250/0.667)K$ **T_{o2}=375 K**

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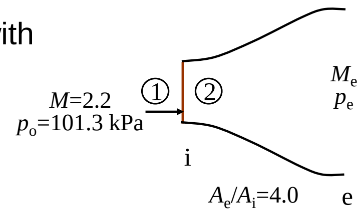
Examples: Problem 2

- **Given**: Diverging nozzle with exit/inlet area ratio of 4.0
  - normal shock stands at entrance
  - just before entrance He, Mach 2.2 with stagnation pressure of 101.3 kPa



- **Find**:
  1. Mach number at nozzle exit
  2. (static) pressure at exit
- **Assume**: He is tpg/cpg, γ=5/3 for T range (unknown)

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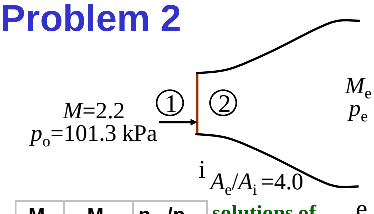
Solution: Problem 2

- **Analysis:**
  - M_e get from area ratio



isentropic 2→e

$$\frac{A_e}{A_i} = \frac{(A_e/A^*)_{M_e}}{(A_i/A^*)_{M_2}} = 4.0$$

– but need M_2

M_1=M_i=2.2 $\overset{\text{V.C11}}{\Rightarrow}$ M_2=0.581

M_1	M_2	p_{o2}/p_{o1}
2.20	0.5813	0.6860

solutions of V.C11,13

$$\left.\frac{A_e}{A^*}\right|_{M_e} = 4\left.\frac{A_i}{A^*}\right|_{0.581} = 4\times1.20 = 4.80 \Rightarrow \boxed{M_e} = \begin{cases} \boxed{0.118} \\ 3.73 \end{cases}$$

V.B2

V.B2,A6

M	A/A*	p/p$_o$
0.118	4.799	0.9884
0.581	1.198	0.0906
3.732	4.799	0.0132

– p_e:

$$p_e = p_{oe}\frac{p_e}{p_{oe}} = \left(\frac{p_{o2}}{p_{o1}}p_{o1}\right)\left.\frac{p}{p_o}\right|_{M_e=0.118}$$

V.C13 V.A6

$$p_e = (0.686\times101.3kPa)0.988$$

$\boxed{p_e\text{=68.7 kPa}}$

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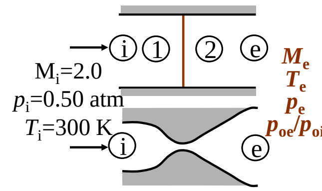
Examples: Problem 3

- **Given**: Two engine inlets, one a straight tube, the other a converging-diverging diffuser
  - both with M=2.0 at their inlets, T_i=300.K, p_i=0.50 atm
  - both slow flow down to same M_e
    - CD diffuser: reversible flow
    - straight diffuser: with shock



- **Find**: compare performance of these 2 diffusers
  - M_e, T_e, p_e, p_{oe}/p_{oi}
- **Assumptions**:
  - air is cpg/tpg with γ=1.4
  - adiabatic diffusers
  - reversible in straight diffuser except at shock

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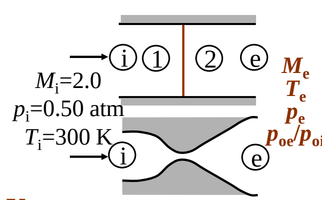
Solution: Problem 3

- **Analysis: "Shock Diffuser"**
  - γ=1.4, solutions of V.C8-11, 13, 17

M_1	M_2	p_2/p_1	T_2/T_1	ρ_2/ρ_1	p_{o2}/p_{o1}	A^*_2/A^*_1
2.00	0.5774	4.500	1.688	2.667	0.7209	1.387



  - $M_e=M_2=$**0.577**
  - $T_e=T_2=(T_2/T_1)T_1=(1.688)300\text{K}=$ **506 K**
  - $p_e=p_2=(p_2/p_1)p_1=(4.50)0.5\text{atm}=$**2.25 atm**; $p_{oe}/p_{oi}=p_{o2}/p_{o1}=$ **0.721**
- **"CD Diffuser"** - isentropic (solutions of V.A5-6, B2)
  - M_e=**0.577**
  - $T_e=\dfrac{\left(T_e/T_o\right)\big|_{0.577}}{\left(T_i/T_o\right)\big|_{2.0}}T_i=\dfrac{0.938}{0.556}\times 300K=$ **506 K** *same (M, T_o const)*
  - $p_e=(T_e/T_i)^{\gamma/\gamma-1}p_i=(506/300)^{3.5}0.5\text{atm}=$**3.12 atm** ← *higher p*
  - p_{oe}/p_{oi} = **1** ← *no p_o loss ⇒ higher mass flowrate or smaller area*

M	A/A*	T/T_o	p/p_o
0.577	1.217	0.9376	0.7980
2.000	1.688	0.5556	0.1278