## Entropy Change with Heat Addition

In the Brayton cycle, we replaced the (actual) combustion process with (ideal) reversible heat transfer, i.e., a reversible heat exchanger. It was stated then that reversible heat addition (with no work) implied a constant pressure process. This handout examines in more detail the conditions required.

Temperature Constraints for Reversible Heat Transfer

We begin by examining a device that adds heat to a flow reversibly, *but does no work.* As illustrated in Fig. 1, imagine heat transfer from one body to another occurring along the adjoining length of two control volumes, one surrounding each body. Let $\delta\dot{Q}$ be the heat transfer rate along an infinitesimal length of the walls. To produce reversible heat transfer, the first condition is that the heat transfer must occur across an infinitesimally small temperature difference, or equivalently, the local wall temperature of the two bodies must equal be equal at any given distance along the walls. Otherwise, the entropy transferred from one body (e.g. $\delta\dot{Q}/T_{wall,1}$ [1]) would not be the same as the entropy transferred to the second body (e.g. $\delta\dot{Q}/T_{wall,2}$).
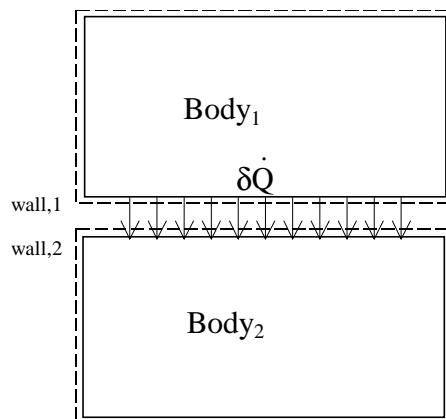
Figure 1. Heat transfer from body 1 to body 2.

In other words, consider two bodies with a finite temperature difference and with $T_{wall,1} > T_{wall,2}$.[2] In this case, the entropy transfer out of body 1 would be less than the entropy transferred into body 2 ($\delta\dot{Q}/T_{wall,1} < \delta\dot{Q}/T_{wall,2}$). This means that entropy must have been produced in the process, thereby making the process **irreversible**. Thus we see that a **<u>reversible heat exchanger must have an infinitesimally small temperature difference between the bodies/fluids exchanging energy</u>**.

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[1]Recall that heat transfer into (or out of) a control volume also transports entropy into (or out of) the control volume.

[2]The condition, $T_{wall,1} < T_{wall,2}$, is impossible since it would require heat transfer from a colder body to a warmer body without doing work.

Pressure Constraints for Reversible Heat Transfer to a Fluid: Low Velocity

Now consider a fluid going through the heat exchanger, exchanging energy with the walls of the device (Fig. 2), but again assuming no work (except flow work) is done. As we just saw, the local wall temperature must equal the temperature of the fluid at a given distance along the device.
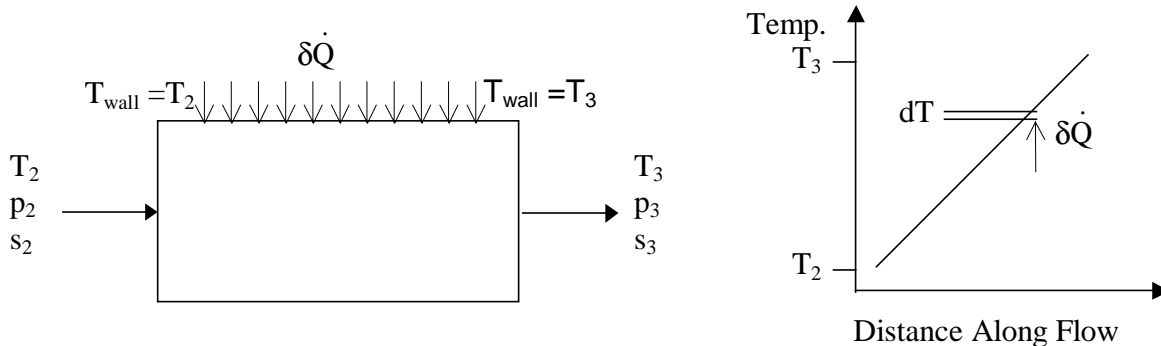
Figure 2. Heat transfer into a fluid in a reversible heat exchanger.

For such a control volume (which again is an idealized model of our combustor), the *conservation equations* assuming a calorically and thermally perfect gas, **with negligible velocity**, are:

Mass: (if this was a combustor, we would be ignoring mass addition due to the fuel)

$$\dot{m}_2 = \dot{m}_3 = \dot{m}$$

Energy: (constant c_p, no work but flow work)

$$\dot{m}h_2 + \dot{Q} = \dot{m}h_3$$

$$\dot{m}h_2 + \int_2^3 \delta\dot{Q} = \dot{m}h_3$$

therefore, $\qquad \int_2^3 \delta\dot{Q} = \dot{m}c_p \int_2^3 dT$

or $\qquad \delta\dot{Q} = \dot{m}c_p dT$

Entropy: (reversible=no entropy production)

$$\dot{m}s_2 + \int_2^3 \frac{\delta\dot{Q}}{T} = \dot{m}s_3$$

$$\int_2^3 \frac{\delta\dot{Q}}{T} = \dot{m}(s_3 - s_2)$$

Replacing the heat transfer term on the left-hand side of this equation with the result from the energy equation, and using our state relation for Δs on the right-hand side, we find

$$\int_2^3 \frac{\dot{m} c_p dT}{T} = \dot{m}\left(c_p \, ln\frac{T_3}{T_2} - R \, ln\frac{p_3}{p_2} \right)$$

and then solving the integral for c_p=const.,

$$\dot{m} c_p \, ln\frac{T_3}{T_2} = \dot{m}\left(c_p \, ln\frac{T_3}{T_2} - R \, ln\frac{p_3}{p_2} \right).$$

$$or \;\; ln\frac{p_3}{p_2} = 0$$

The only way this can be true is for p_3/p_2=1, i.e**., reversible heat addition for a flow with negligible velocity must take place at constant pressure**.

Pressure Constraints for Reversible Heat Transfer to a Fluid: Arbitrary Velocity

Now let us consider the case of reversible heat transfer (no work) to a fluid with **non-negligible velocity**. Again writing the conservation equations:

Mass: (same as above)

Energy: (again constant c_p, no work, but now use stagnation values to include kinetic energy)

$$\dot{m} h_{o2} + \dot{Q} = \dot{m} h_{o3}$$

$$\dot{m} h_{o2} + \int_2^3 \delta\dot{Q} = \dot{m} h_{o3}$$

$$\int_2^3 \delta\dot{Q} = \dot{m}\int_2^3 c_p dT_o$$

$$\delta\dot{Q} = \dot{m} c_p dT_o$$

Entropy: (again no entropy production)

$$\dot{m} s_{o2} + \int_2^3 \frac{\delta\dot{Q}}{T} = \dot{m} s_{o3}$$

$$\int_2^3 \frac{\delta\dot{Q}}{T} = \dot{m}\left(s_{o3} - s_{o2} \right)$$

Again using the energy equation to replace the heat transfer and using the Δs equation of state:

$$\int_2^3 \frac{\dot{m} c_p dT_o}{T} = \dot{m}\left(c_p \, ln\frac{T_{o3}}{T_{o2}} - R \, ln\frac{p_{o3}}{p_{o2}} \right)$$

$$\int_2^3 \frac{\dot{m} c_p T_o}{T} \frac{dT_o}{T_o} = \dot{m}\left(c_p \, ln\frac{T_{o3}}{T_{o2}} - R \, ln\frac{p_{o3}}{p_{o2}} \right)$$

$$\dot{m} c_p \int_2^3 \left(1 + \frac{\gamma-1}{2} M^2 \right)\frac{dT_o}{T_o} = \dot{m}\left(c_p \, ln\frac{T_{o3}}{T_{o2}} - R \, ln\frac{p_{o3}}{p_{o2}} \right)$$

where T_o/T was replaced in the last step with the function of Mach number in parentheses. Again solving the integral, and assuming some average value of M (since M could change as heat is added),

$$\dot{m}c_p\left(1+\frac{\gamma-1}{2}M_{av}^{\ 2}\right)\ln\frac{T_{o3}}{T_{o2}}=\dot{m}\left(c_p\ln\frac{T_{o3}}{T_{o2}}-R\ln\frac{p_{o3}}{p_{o2}}\right).$$

We see that if M_{av} is not zero, $p_{o3}\neq p_{o2}$. The general result for p_{o3}/p_{o2} is found by solving the above equation.

$$\frac{R}{c_p}\ln\frac{p_{o3}}{p_{o2}}=-\left(\frac{\gamma-1}{2}M_{av}^{\ 2}\right)\ln\frac{T_{o3}}{T_{o2}}$$

$$\ln\frac{p_{o3}}{p_{o2}}=\ln\frac{T_{o3}}{T_{o2}}^{-\left(\frac{\gamma-1}{2}M_{av}^{\ 2}\right)\frac{c_p}{R}}=\ln\frac{T_{o3}}{T_{o2}}^{-\left(\frac{\gamma-1}{2}M_{av}^{\ 2}\right)\frac{\gamma}{\gamma-1}}$$

$$\frac{p_{o3}}{p_{o2}}=\left(\frac{T_{o2}}{T_{o3}}\right)^{\frac{\gamma}{2}M_{av}^{\ 2}}$$

where it is clear that $\mathbf{p_{o3}/p_{o2}=1}$ **only if** a) $T_{o3}/T_{o2}=1$, i.e., no heat addition, or b) $M_{av}=0$, i.e., heat addition at negligible velocity. Otherwise $p_{o3}/p_{o2}<1$, i.e**., <u>reversible</u> heat <u>addition</u> in high speed flows leads to a loss in stagnation pressure**.