Review of Compressible Flow

What is a compressible flow?

Incompressible Flow

- In continuum fluid mechanics, often start by considering conservation/transport equations, e.g.,
  - mass conservation ("continuity equation")
  - momentum conservation
- For *incompressible flow*, these two conservation equations are sufficient to define the behavior of a fluid
  - but this requires an isodensity constraint (ρ=*constant*)
    - e.g., leads to Bernoulli's equation (*only valid for incompressible flow along a streamline*)
      - $p_o = p + \frac{1}{2} \rho u^2$ *pressure and velocity (changes) in a flow are linked*
    - assuming constant density is reasonable for nearly incompressible fluid, e.g., many liquids

Gas Compressibility

- What about a gas flow?
- Gases are definitely compressible
    - $\rho = \rho(p, T)$
  - compressibility coefficients for near-tpg (thermally perfect gas) at SATP
    $\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big|_p \sim -O\left(3\times10^{-3}\frac{1}{K}\right); \kappa = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\Big|_T \sim O\left(10^{-5}\frac{m^2}{N}\right)$
  - so 10K increase in T will lower density by ~3%, as will a 300 Pa drop in p
- When can we treat a gas **flow** as incompressible?
  - when a fluid element only undergoes small pressure and temperature changes as it travels along the flow!!

Compressible Flow

- Why would T change in a flow?
  - one reason: heating (Q)
  - another reason: conversion of flow kinetic energy ($\frac{1}{2}u^2$) to/from thermal energy: **internal energy**(e)/**enthalpy**(h)
- We will **define** a ***compressible flow*** as one where the changes in kinetic energy have a non-negligible impact on temperature, and thus density (and pressure)
- To model/analyze compressible flow
  - must include energy conservation equation (in addition to mass and momentum conservation)

Mach Number and Sound Speed

- To gauge the relative importance of a flow's kinetic energy, it is often useful to define a dimensionless quantity, the Mach number

named for Ernst Mach, Austrian ~1870

(I.B.1) $$M \equiv \frac{u}{a}$$

u ← flow speed; a ← speed of sound

  - e.g, common to say flow is compressible for $M > 0.3$

- Here, speed of sound of a simple compressible substance is defined as

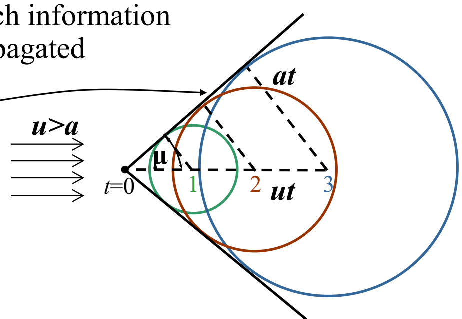
(I.B.2) $$a^2 \equiv \frac{1}{\rho\beta} = \frac{\gamma}{\rho\kappa} = \gamma \left.\frac{\partial p}{\partial \rho}\right|_T$$

specific heat ratio $\gamma \equiv c_p/c_v$

β: isentropic compressibility; κ: isothermal compressibility

Mach Wave and Mach Angle

- In supersonic flow, there is a region where pressure "disturbances" have had an effect (been "heard")
- **Weak pressure disturbances propagate as sound waves**
  - for **weak disturbance** at a point in supersonic flow, get conical region within which information about disturbance has propagated
  - region delineated by a **Mach line (Mach wave)**
  - angle between Mach line and flow direction is **Mach angle, μ**



(I.B.3) $$\mu = \sin^{-1}\left(\frac{at}{ut}\right) = \sin^{-1}\left(\frac{1}{M}\right)$$

sound waves can't propagate upstream in supersonic flow

Review of Compressible Flow

What are the governing equations?

Compressible Flow Equations

- To understand and model compressible flow, we need

1. **Conservation equations** for open systems
   (also called transport equations)
   - mass, momentum, **and** energy
   - also species (if reacting)
   - can include diffusion processes
     or if neglect them called "inviscid" versions
2. **State equations** (if equilibrium conditions apply)
   - p-v-T relation (leads to compressibility coefficients)
   - caloric relations (give specific heats, c_v and c_p)
   - entropic (can get from p-v-T and caloric)

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Inviscid Conservation/Transport Eqns.

$$\frac{D?}{Dt} = \frac{\partial ?}{\partial t} + u_j \frac{\partial ?}{\partial x_j}$$

substantial derivative

- **Mass/continuity** $\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0$
- **Momentum** $\rho \frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \rho F_j$ (body force)
- **Energy** $\rho \frac{Dh_o}{Dt} = \frac{\partial p}{\partial t} + \rho F_j u_j + \dot{q}$ ($\dot{q}$: volumetric heating, e.g., radiation)

  stagnation enthalpy, $h_o \equiv h_{mix} + ½ u^2$
- **Species** $\frac{DY_i}{Dt} = \frac{\dot{w}_i}{\rho}$ ($\dot{w}_i$: net mass production rate of species i) **(I.B.4)**

all molecular diffusion terms neglected in these equations

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State Equations - Examples

- ***p-v-T*** $p/\rho = RT$ thermally perfect (ideal) gas - **TPG**

  $(p + a\rho^2)(1/\rho - b) = RT$ Van der Waals state eq: a,b constants
- **Caloric** e, h with $h = e + pv = e + p/\rho$
  - *single (or non-reacting) tpg* $e = e(T)$; $h = e(T) + RT$

    $c_v = de/dT$ $c_p = c_v + R$

    $c_p = dh/dT$ **if also calorically perfect gas – CPG** ⇒ c_p, c_v, γ=constants
  - *if reacting tpg, R≠const.* $h_{mix} = \sum Y_i h_i = \sum Y_i e_i + \sum (Y_i R_i) T$

    $= e_{mix}(p,T) + R_{mix}(p,T)\, T$

    $= h_{mix}(p,T)$ *if non-reacting tpg*
- **Speed sound** $a^2 = \gamma\,(\partial p/\partial \rho)_T$ $a^2 = \gamma RT$ **(I.B.5)**

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State Equations - Examples

- **Entropic** $Tds = dh - vdp$
  - *if tpg* $ds = c_p dT/T - Rdp/p$
  - *if also cpg* $\Delta s_{12} = c_p ln(T_2/T_1) - Rln(p_2/p_1)$
  - so if tpg/cpg fluid element undergoes isentropic process

$$ln(p_2/p_1)^R = ln(T_2/T_1)^{c_p}$$

$$p_2/p_1 = (T_2/T_1)^{c_p/R} = (T_2/T_1)^{\gamma/(\gamma-1)}$$

with $p=\rho RT$ $$\rho_2/\rho_1 = (T_2/T_1)^{c_v/R} = (T_2/T_1)^{1/(\gamma-1)}$$

$\frac{c_p}{R} > \frac{c_v}{R} > 1$ **changing T in isentropic flow**
will also change p in same direction *(and by more)*
will also change ρ in same direction
(by more than T but less than p)

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Stagnation Properties

- Properties that would (theoretically) be achieved if a fluid element was brought to rest *in a chosen reference frame*:
  1) with no external work,
  2) adiabatically + 3) *reversibly?*

expressions below for tpg+cpg

- **Stagnation Temperature** $c_p(T_o - T) = \frac{u^2}{2}$ (Bulk KE) $\Rightarrow \frac{T_o}{T} = 1 + \frac{u^2/2}{c_p T}$ $\frac{1}{c_p T} = \frac{\gamma-1}{\gamma RT}$
  - from energy conservation: *no work* but flow work and **adiabatic**
  - $\Rightarrow$ **T_o** (and **h_o**) **constant for adiabatic flow**

(I.B.6) $$\boxed{\frac{T_o}{T} = 1 + \frac{\gamma-1}{2}M^2}$$

$M^2 \propto$ kinetic/thermal energy of flow

- **Stagnation Pressure** $p_o/p = (T_o/T)^{\gamma/\gamma-1}$
  - for adiabatic + **reversible** =isentropic process **(Δs=0)**
  - $\Rightarrow$**p_o** (and **s_o**) **constant if *also* reversible**

(I.B.7) $$\boxed{\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}}$$

$\rho = p/RT$

- **Stagnation Density**

(I.B.8) $$\boxed{\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/\gamma-1}}$$

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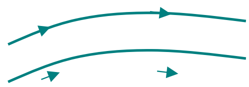
Adiabatic Flow Ellipse

- One way to look at M effects on flow properties
- **Energy equation**

$$h_o = h + \frac{u^2}{2} = const$$

- **Stagnation T_o** also constant



adiabatic/no work stream tube

tpg/cpg

$$T_o = T + \frac{\gamma - 1}{2}\frac{u^2}{\gamma R} = \text{const}$$

$$\frac{2}{\gamma - 1}\gamma RT + u^2 = \text{const}$$

$$\frac{2}{\gamma - 1}a^2 + u^2 = u_{\max}^2 = \frac{2}{\gamma - 1}a_o^2$$

Stagnation speed of sound (no kinetic energy left, u=0)

Maximum velocity possible (no thermal energy left, T=0)

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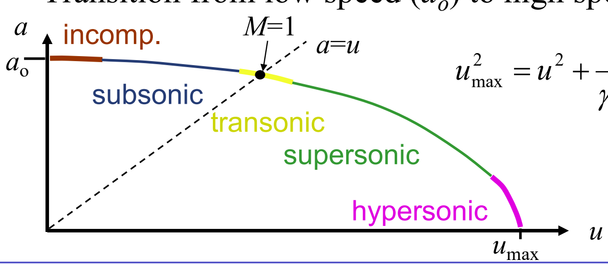
Adiabatic Flow Ellipse: Flow Regimes

- Transition from low speed (a_o) to high speed ($u_{\max}$)



$$u_{\max}^2 = u^2 + \frac{2}{\gamma - 1}a^2 = \frac{2}{\gamma - 1}a_o^2$$

Regime	Description/Interpretation
incomp.	$u \ll a$, $da \ll du$, little change in $a(T)$
subsonic	$u \lesssim a$, M changes primarily to changes in u
transonic	$\lvert u-a \rvert \ll u,a$
supersonic	$u>a$, M changes through substantial changes in u and $a(T)$
hypersonic	$u \gg a$, $du \ll da$, M change mostly due to $a(T)$ changes

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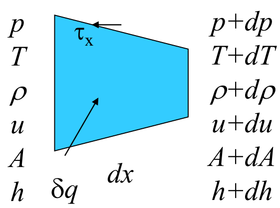
Review of Compressible Flow

How do the compressible flow equations simplify for quasi-1D flow for a tpg/cpg?

Quasi-1D, Steady Equations

- Examine control volume with differential length *dx*, in steady, nonreacting flow, no body forces
  - allow heat transfer and shear stress
  - **quasi-1D** ≡ variations only in *x* direction



Mass $$\frac{d\rho}{\rho}+\frac{1}{2}\frac{du^2}{u^2}+\frac{dA}{A}=0$$

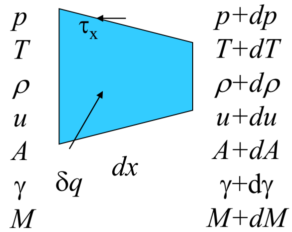
valid only for *dA*/*dx* small

Momentum (shear stress, normal stress, momentum change)
$$\frac{\tau_x}{p}\frac{L_p}{A}dx+\frac{dp}{p}+\frac{1}{2}\frac{u^2}{p/\rho}\frac{du^2}{u^2}=0$$

Energy (heat addition, KE change, thermal energy change)
$$\frac{\delta q}{RT}-\frac{1}{2}\frac{u^2}{RT}\frac{du^2}{u^2}-\frac{dh}{RT}=0$$

Compressible Ideal Gas Equations

- In addition, limit to non-reacting **tpg** (non-react.⇒R=const.)



Mass $$\frac{d\rho}{\rho}+\frac{1}{2}\frac{du^2}{u^2}+\frac{dA}{A}=0$$

Momentum $$\frac{\tau_x}{p}\frac{L_p}{A}dx+\frac{dp}{p}+\frac{\gamma}{2}M^2\frac{du^2}{u^2}=0$$

Energy $$\frac{\delta q}{c_pT}-\frac{(\gamma-1)}{2}M^2\frac{du^2}{u^2}-\frac{dT}{T}=0$$

Ideal Gas Eq. State $$\frac{dp}{p}-\frac{d\rho}{\rho}-\frac{dT}{T}=0$$

Mach Number $$\frac{dM^2}{M^2}-\frac{du^2}{u^2}+\frac{dT}{T}+\frac{d\gamma}{\gamma}=0$$

Mach Number Changes

- Combine conservation/state equations
  - can algebraically show **if tpg+cpg**

$f=\frac{4\tau_x}{1/2\,\rho v^2}$ $D=\frac{4A}{L_p}$

friction factor- from τ_x

(I.B.9) $$\frac{dM^2}{M^2}=\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\left\{\frac{\delta q}{c_pT_o}\left(1+\gamma M^2\right)+\gamma M^2\frac{fdx}{D}-2\frac{dA}{A}\right\}$$

- So we have three ways to change M of flow
  - **area change (*dA*)**: e.g., converging-diverging nozzles
  - **friction**: $f>0$, same effect as $-dA$
  - **heat transfer**: heating, $\delta q>0$, like $-dA$; cooling, $\delta q<0$, like $+dA$

Mach Number Changes

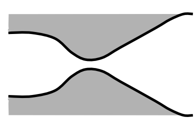
$$\frac{dM^2}{M^2} = \frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\left\{\frac{\delta q}{c_p T_o}\left(1+\gamma M^2\right)+\gamma M^2 \frac{fdx}{D} - 2\frac{dA}{A}\right\}$$

- Subsonic flow (M<1): $1-M^2 > 0$
  - friction, heating, converging area ⇒ increase M (dM>0)
  - cooling, diverging area ⇒ decrease M (dM<0)
- Supersonic flow (M>1): $1-M^2 < 0$
  - friction, heating, converging area ⇒ decrease M (dM<0)
  - cooling, diverging area ⇒ increase M (dM>0)

Sonic Condition: Throat Requirement

$$\frac{dM^2}{M^2} = \frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\left\{\frac{\delta q}{c_p T_o}\left(1+\gamma M^2\right)+\gamma M^2 \frac{fdx}{D} - 2\frac{dA}{A}\right\}$$

- Consider smooth transition subsonic⇔supersonic flow
  - must go through M=1 (sonic condition, denoted by "*")
- As $M \to 1$, $1-M^2 \to 0$, need { } term to approach 0
- For **isentropic flow**,
  - sonic condition is dA=0, **A^* is a minimum area ("throat")** in the flow



- With friction and/or heating, need dA>0
  - A^* occurs in a diverging area region
- For cooling (no friction), need dA<0
  - A^* occurs in a converging area region

Mach Number Relations

- For tpg/cpg + steady, get equations for each TD property change as function of *Mach number* change

$$\frac{dT_o}{T_o} = \frac{\delta q}{c_p T_o} \qquad \frac{d\rho}{\rho} = -\frac{1}{2}\frac{du^2}{u^2} - \frac{dA}{A} \qquad \frac{dp}{p} = -\frac{\gamma}{2}M^2\left(\frac{du^2}{u^2} + \frac{fdx}{D}\right)$$

$$\frac{dT}{T} = \frac{dT_o}{T_o} - \frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2} \qquad \frac{dp_o}{p_o} = \frac{dp}{p} + \frac{\frac{\gamma}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2}$$

$$\frac{du^2}{u^2} = \frac{dT_o}{T_o} + \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM^2}{M^2} \qquad \frac{ds}{R} = \frac{\gamma}{\gamma-1}\frac{dT_o}{T_o} - \frac{dp_o}{p_o}$$

- Analytic solutions to these differential eqns. exist if ONLY
  area change : **Isentropic** (e.g., nozzle) flow
  OR friction : **Fanno** flow
  OR heat transfer : **Rayleigh** flow

Review of Compressible Flow

How does quasi-1D, steady isentropic flow behave for a tpg/cpg?

Isentropic Flow: Property Changes

- For tpg/cpg, T_o **and** p_o **(and** ρ_o**) are constant** in isentropic flow

$$\frac{T_o}{T}=1+\frac{\gamma-1}{2}M^2 \;<\; \frac{\rho_o}{\rho}=\left(1+\frac{\gamma-1}{2}M^2\right)^{1/\gamma-1} \;<\; \frac{p_o}{p}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$

- If M increases (an **expansion**)
  - T, ρ, and p will decrease
- If M decreases (a **compression**)
  - T, ρ, and p will increase
- T will change least, p will change most

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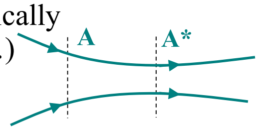
Area Ratio

- For isentropic flow, look at effect of area change on M by comparing A at any point to area at sonic point (A^*)
  - * refers to properties of a flow if isentropically accel./decel. to M=1 sonic (e.g. ρ^*, T^*,…)
  - alternative to stagnation as ref. state



- Use **mass conservation** to find relation

$$\rho u A = \rho^* u^* A^*$$

$$\frac{A}{A^*}=\frac{\rho^*}{\rho}\frac{u^*}{u}=\frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{u^*}{a^*}\frac{a^*}{a}\frac{a}{u}$$

($u^* = a^*$; $\frac{a}{u} = \frac{1}{M}$)

$$\frac{\rho_o}{\rho}=\left(1+\frac{\gamma-1}{2}M^2\right)^{1/\gamma-1} \qquad \frac{\rho_o}{\rho^*}=\left(1+\frac{\gamma-1}{2}\right)^{1/\gamma-1}$$

$$\frac{a^*}{a}=\sqrt{\frac{T^*}{T}}=\sqrt{\frac{T^*}{T_o}\frac{T_o}{T}} \text{ with } \frac{T_o}{T}=1+\frac{\gamma-1}{2}M^2, \quad \frac{T_o}{T^*}=1+\frac{\gamma-1}{2}$$

$$\boxed{\frac{A}{A^*}=\frac{1}{M}\left\{\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right\}^{\gamma+1/2(\gamma-1)}}$$

(I.B.10)
for tpg/cpg

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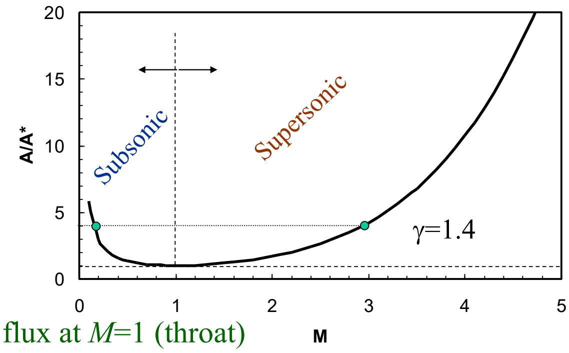
Area Ratio Results

- Two (isentropic) solutions for a given A/A^*
  - one subsonic
  - one supersonic
- $A \geq A^*$ always
- Acceleration to high M requires large A/A^*
- $\dfrac{\dot{m}/A}{(\dot{m}/A)^*} = \dfrac{\rho u}{\rho^* u^*} = \dfrac{A^*}{A}$
  - maximum mass flux at M=1 (throat)

$$\frac{A}{A^*} = \frac{1}{M}\left\{\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right\}^{\gamma+1/2(\gamma-1)}$$



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Mass Flux and Stagnation Properties

- Examine mass flux in terms of stagnation conditions

$$\frac{\dot{m}}{A} = \rho u = \frac{p}{RT}Ma = \frac{p}{RT}M\sqrt{\gamma RT}$$

tpg

tpg, cpg

$$\frac{T_o}{T} = 1+\frac{\gamma-1}{2}M^2$$

$$\frac{p_o}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/\gamma-1}$$

(I.B.11)

$$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}}\frac{\sqrt{\gamma}M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma+1/2(\gamma-1)}}$$

or $\dfrac{\dot{m}}{A} = \dfrac{p_o}{\sqrt{RT_o}}f(\gamma,M)$

for given isentropic flow, all stagnation (and sonic) properties constant, so is mass flow rate

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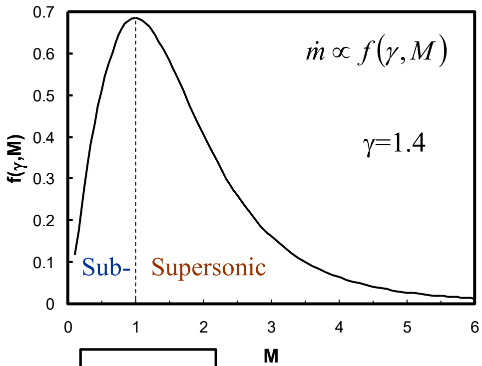
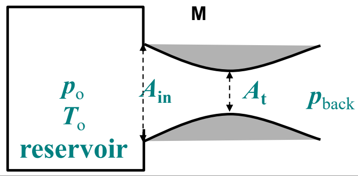
Choked Flow

$$\dot{m} = A\frac{p_o}{\sqrt{RT_o}}f(\gamma,M)$$

- For fixed stagnation properties and flow area
  - max $\dot{m}$ at $M = 1$
- For nozzle with fixed stagnation properties and sonic throat
  - can't alter mass flow rate by changing downstream pressure boundary condition (*back* pressure)

  $\Rightarrow$ **Choked Flow**





Choked Mass Flowrate

- Sonic condition (*at throat*) $\Rightarrow$ maximum flow rate

$$\dot{m}_{max} = A^*\frac{p_o}{\sqrt{RT_o}}\underbrace{\sqrt{\gamma}\left(1+\frac{\gamma-1}{2}\right)^{\gamma+1/2(1-\gamma)}}_{f(\gamma,1)}$$

(I.B.12) **for tpg/cpg**

- To **increase** mass flowrate
  - **increase A^*** ($=A_t$ throat size)
  - **increase p_o** and/or **decrease T_o** (either increases ρ_o)
- $f(\gamma,1)$ typically near 0.7

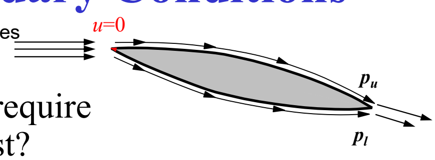
$$\dot{m}_{max} \approx 0.7\frac{p_o}{\sqrt{RT_o}}A_{throat} \qquad f(\gamma,1) = \begin{cases} 0.726 & \gamma = 5/3 \\ 0.685 & \gamma = 1.4 \\ 0.667 & \gamma = 1.3 \end{cases}$$

"rule of thumb" for **choked** gas flows

Review of Compressible Flow

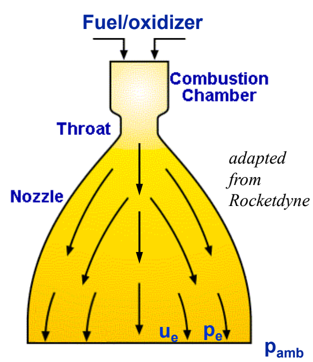
How does a supersonic flow adjust to boundary conditions?

Supersonic Boundary Conditions



- What are the boundary conditions (BC) that can require a supersonic flow to adjust?
  - surface requiring change in flow direction ("turn")
  - stagnation (u=0)
  - pressure matching
- So 2 kinds BC
  - *velocity, pressure*

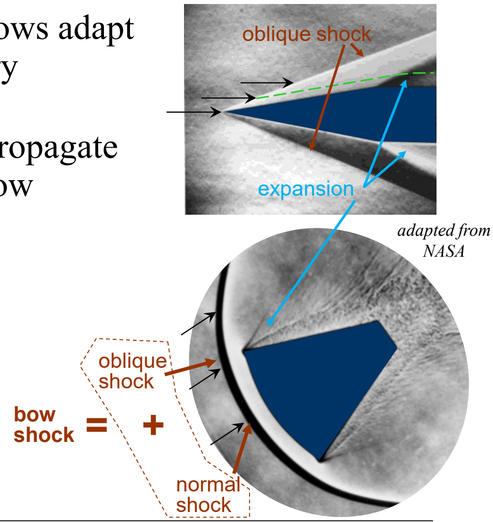
adapted from The Exploration Company



adapted from Rocketdyne

Waves

- How do supersonic flows adapt to changes in boundary conditions (BC)?
- Through waves that propagate BC into supersonic flow
  - "strong waves"
    shocks
    - *compression only*
  - "weak waves"
    sound waves
    - *expansion or compression*



Review of Waves

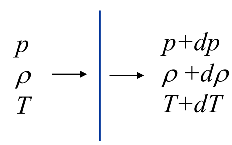
- From this point on, restrict analysis to flows that are
  - steady
  - adiabatic
  - inviscid (except within shocks)
  - 1-d or 2-d
- Review
  - normal shocks
  - oblique shocks
  - Prandtl-Meyer expansions and compressions

Review of Compressible Flow

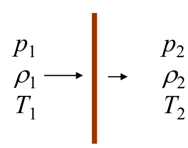
What is a normal shock and how do flow properties change?

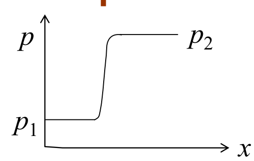
Sound Waves vs Shock Waves

- **Sound waves**
  - weak, $d\rho/\rho \rightarrow 0$
  - reversible and isentropic



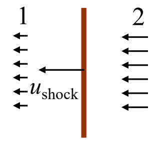
- **Shocks**
  - *strong compressions*, $\rho_2 > \rho_1$
  - "thin" regions (~µm's at STP)
    - changes in fluid properties are *nearly discontinuous*
  - rapid change in properties due to internal viscous stresses
    - so *irreversible*
  - excluding radiation, *adiabatic*
    - so **nonisentropic**



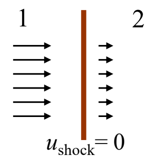


Normal Shock Waves

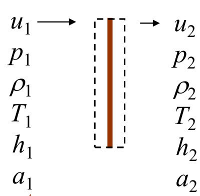
- For **normal shock**
  - wave perpendicular to flow (propagation) direction
- Shock is nonequilibrium process internally, but assume
  - flow *before shock* (1) is **in equilibrium**
  - flow *after shock* (2) is **in equilibrium**



- Easiest way to analyze shock is with control volume in shock's reference frame (moving with shock)
  - equations first studied by Rankine (~1870) and Hugoniot (~1877)



Governing Equations

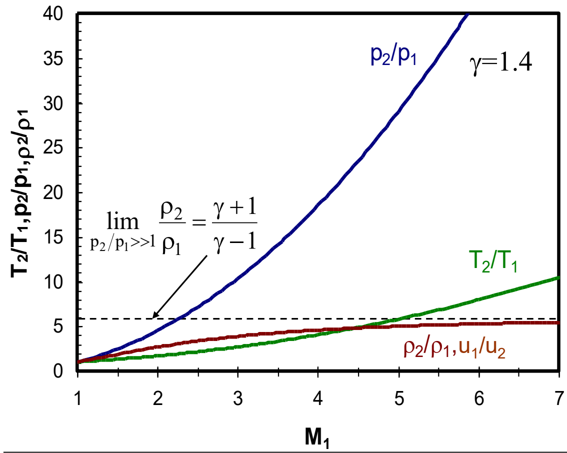


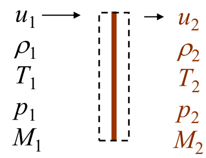
- Conservation equations become
  - mass $\dot{m}/A = \rho_1 u_1 = \rho_2 u_2$
  - momentum $p_1 A_1 + \dot{m} u_1 = p_2 A_2 + \dot{m} u_2$
    $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 = constant$
  - energy $h_1 + u_1^2/2 = h_2 + u_2^2/2 = h_o = constant$
  - combine $h_2 - h_1 = ½\,(p_2 - p_1)(1/\rho_1 + 1/\rho_2)$ **Shock Hugoniot Equation** **(I.B.13)**
- For **tpg/cpg** $\frac{\gamma}{\gamma-1}(p_2/\rho_2 + p_1/\rho_1) = ½\,(p_2 - p_1)(1/\rho_1 + 1/\rho_2)$
  - so density ratio $\rho_2/\rho_1 = \left(1 + \frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}\right) \Big/ \left(\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}\right)$
  - also $T_{o2} = T_{o1}$ $p_2/p_1 = [2\gamma/(\gamma+1)]M_1^2 - (\gamma-1)/(\gamma+1)$

$$M_2^2 = \left(M_1^2 + \frac{2}{\gamma-1}\right) \Big/ \left(\frac{2\gamma}{\gamma-1}M_1^2 - 1\right) \qquad \frac{T_2}{T_1} = \left(1 + \frac{\gamma-1}{2}M_1^2\right) \Big/ \left(1 + \frac{\gamma-1}{2}M_2^2\right)$$

Normal Shock Property Changes

- Based on tpg/cpg eqn's. using shock fixed reference frame





- T, p and ρ **increase**, u **decreases**
- p increase across normal shock is **greatest static property change**
- Density ratio and velocity ratio approach **limit**
- Also **M_2<1, post-shock flow subsonic**

Which of these still hold for moving shock?

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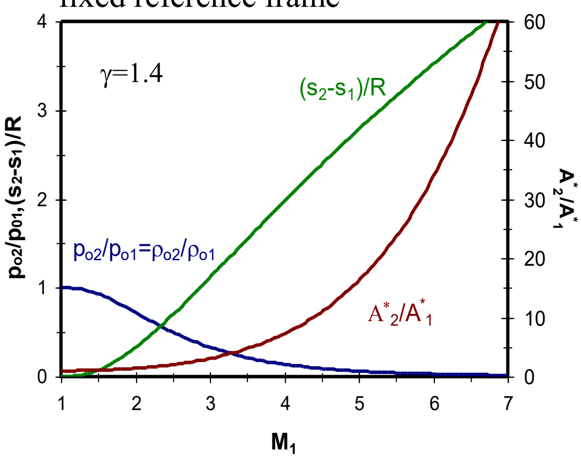
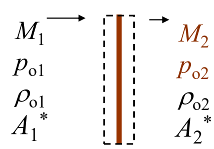
Normal Shock Property Changes

- Based on tpg/cpg eqn's. using shock fixed reference frame





- T_o = *constant*
- p_o *and* ρ_o drop
- Entropy increases
- Sonic area increases
  - larger throat required after shock to reach sonic flow (same mass flowrate, lower p_o)

$$\dot{m} \propto A^* p_o$$

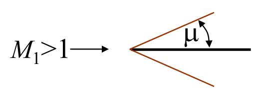
Which of these still hold for moving shock?

Review of Compressible Flow

What is an oblique shock and how do flow properties change?

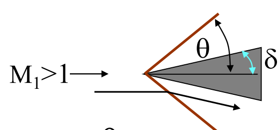
Oblique Shock Waves

- Recall Mach wave
  - consider infinitely thin body
  - no flow turn required
  - infinitessimal wave

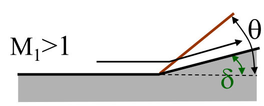


$\mu = \sin^{-1}(1/M_1)$

- Oblique shock
  - example, finite-sized wedge, half-angle, δ, in supersonic flow
  - flow must make sudden, finite turn and compress (flow area reduced)
  - if shock can be attached to tip ⇒ oblique shock at angle θ
  - similar for sharp concave corner



$\theta > \mu$



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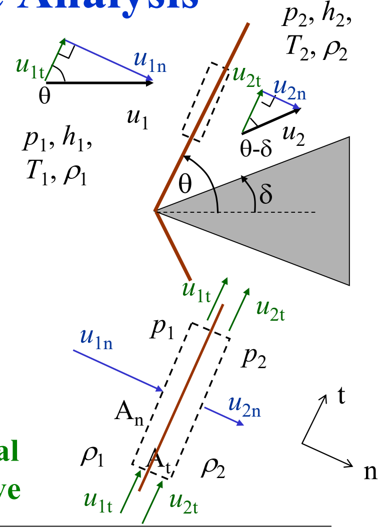
Control Volume Analysis

- Pick CV along shock, divide velocity into two components
  - one tangent to shock, $\boldsymbol{u_t}$
  - one normal to shock, $\boldsymbol{u_n}$
- Result of conservation equations
  - mass $\rho_1 u_{1n} = \rho_2 u_{2n}$
  - mom. $u_{1t} = u_{2t}$
    $p_1 + \rho_1 u_{1n}^2 = p_2 + \rho_2 u_{2n}^2$
  - energy $h_1 + u_{1n}^2/2 = h_2 + u_{2n}^2/2$
- So **oblique shock** acts like **normal shock** in direction **normal to wave**



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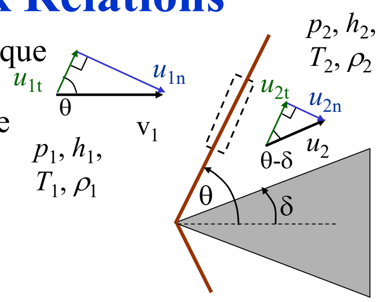
Oblique Shock Relations

- To find conditions across oblique shock, can use normal shock relations, but for tpg/cpg make replacements

  $M_1 \rightarrow M_1 \sin\theta = M_{1n}$
  $M_2 \rightarrow M_2 \sin(\theta-\delta) = M_{2n}$
- Except
  - still have $T_{o1} = T_{o2}$
  - and p_{o2} based on p_2 and M_2
- Shock angle and turn angle related



(I.B.14) $$\tan\delta = \frac{(2/\tan\theta)(M_1^2\sin^2\theta - 1)}{M_1^2(\gamma + \cos 2\theta) + 2}$$ quadratic equation in M_1 and θ

$\Rightarrow$ ***2 solutions for a given (M_1,δ)***

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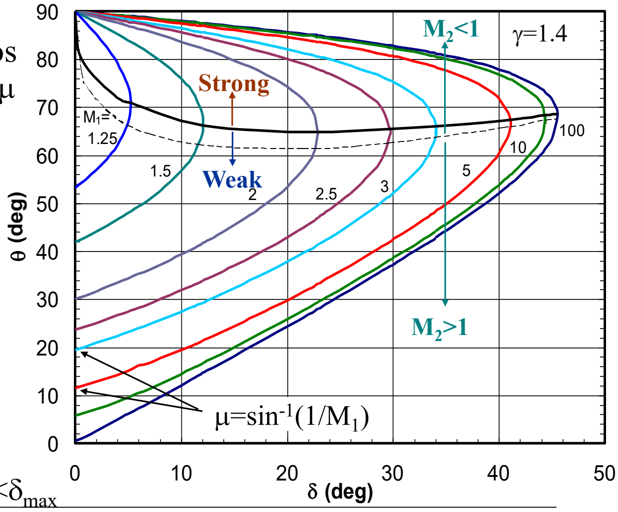
Oblique Shock Chart

- **Weak shocks**
  - lower M_{1n}, ratios
  - smaller θ, $\theta_{min}=\mu$
  - usually $M_2>1$
- **Strong shocks**
  - $\theta_{max}=90°$ (normal shock)
  - always $M_2<1$
- **Both** for δ=0
  - no turn for normal shock or Mach wave
- $\delta_{max}(M_1)$
  - oblique shocks only exist for $\delta<\delta_{max}$



Review of Compressible Flow

What are Prandtl-Meyer fans and how do flow properties change?

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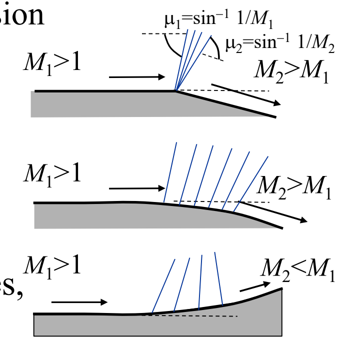
Expansions and Smooth Compressions

- For oblique shocks, we examined supersonic flow over **sharp** concave corners/turns
  – required sudden finite compression
- What happens if:
  1) turn is convex (expansion)?
  2) turn is gradual? (convex or concave)
- Get series of infinitesimal waves, if isentropic ⇒ **Mach waves**
  – **Prandtl-Meyer "fan"**



what happens when weak compression waves converge?

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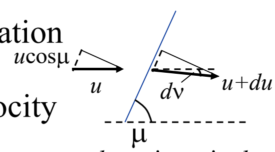
Prandtl Meyer Wave

- **Solution approach**
  – apply mass and momentum conservation to differential CV (wave aligned)
  – result is no change in tangential velocity



$$u\cos\mu = (u+du)\cos(\mu+d\nu)$$

$$= \cos\mu\cos d\nu - \sin\mu\sin d\nu$$
$$= \cos\mu - d\nu\sin\mu$$

(I.B.15) $$\frac{du}{u} = \frac{\sin\mu}{\cos\mu}d\nu = \sqrt{\frac{\sin^2\mu}{1-\sin^2\mu}}d\nu$$

Relates velocity change to turn angle

  – for (isentropic) Mach wave, Mach angle given by $\sin\mu = a/u$

Prandtl Meyer function

$$\frac{du}{u} = \sqrt{\frac{(a/u)^2}{1-(a/u)^2}}d\nu \Rightarrow \boxed{\frac{du}{u} = \sqrt{\frac{1}{M^2-1}}d\nu}$$ (I.B.16)

for tpg/cpg

$$d\nu = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

⇒ turn function of only M for tpg/cpg

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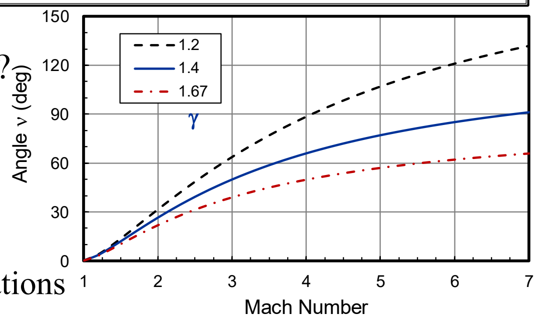
PM Results: tpg/cpg

- To get overall change across PM fan, integrate
  $\delta = \nu_2 - \nu_1 = \int d\nu$ **(I.B.17)**
- For tpg/cpg
  – using ν=0 at M=1

$$\nu_2 - \nu_1 = \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1}\right]_{M_1}^{M_2}$$



- Properties changes?
  – reversible + adiabatic
  – T_o, p_o, ρ_o, s constant
  – T, p, ρ : isen. relations

Compressible Flow -47

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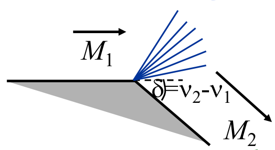
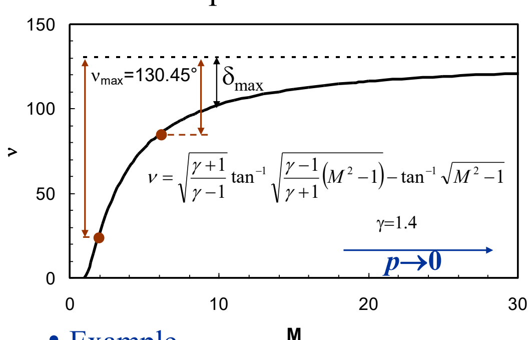
Maximum Prandtl Meyer Turn Angle

- Examine plot of ν as function of M



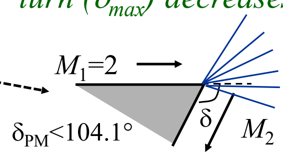


- *as M increases, reach* ***maximum turn angle*** *(ν_{max}~130.5° for γ=1.4)*
- *so as M_1 increases, max. angle flow can turn (δ_{max}) decreases* **why?**

- Example

$M_1 = 2 \Rightarrow \delta_{max} = 130.45^\circ - 26.38^\circ = 104.1^\circ$

$M_1 = 6 \Rightarrow \delta_{max} = 130.45^\circ - 84.96^\circ = 45.5^\circ$



Compressible Flow -48

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Review of Compressible Flow

Examples of aerospace applications with supersonic flowfields containing shocks and expansions

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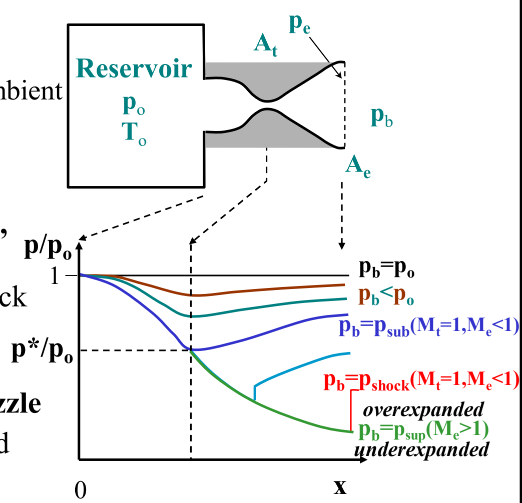
CD (Laval) Nozzle

- For converging-diverging nozzle
  - back pressure p_b (e.g., ambient pressure) is downstream boundary condition
- If $p_b/p_o < p_{sub}/p_o$, choked flow (**sonic throat, M_t=1**)
- Lowering p_b/p_o moves shock from throat to exit
- If $p_b/p_o < p_{shock}/p_o$, **M_e>1** and **isentropic flow in nozzle**
- $p_b = p_{sup}$, perfectly expanded exhaust



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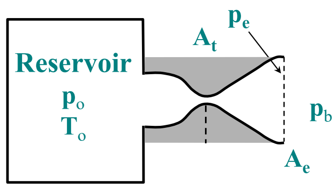
CD (Laval) Nozzle

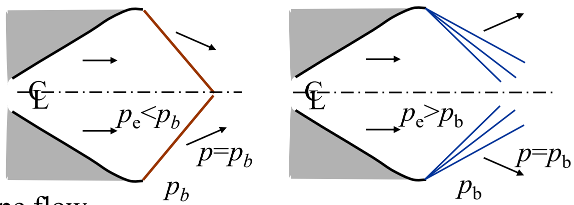
- What happens if under- or over-expanded?
- **Underexpanded**
  - pressure BC at exit requires supersonic expansion

  ⇒ Prandtl-Meyer type flow



- **Overexpanded**
  - pressure BC requires supersonic compression process

  ⇒ oblique shock type flow



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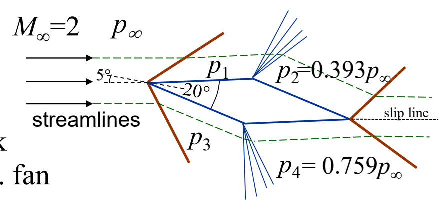
Diamond Airfoil

- 20° diamond (sharp corners) airfoil in Mach 2 air flow
  - assuming air is tpg/cpg with γ=1.4



- **Flow features**
  - upper surface
    - 5° upward = obl. shock
      20° down = PM expan. fan
  - lower surface
    - 15° downward = obl. shock
      20° up = PM expan. fan
  - at trailing edge, must match p and velocity direction = 2 obl. shocks in this case (upper stronger)
    - trailing edge flow close to 0° here, not general result

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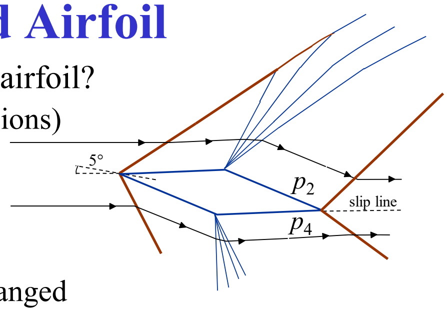
Diamond Airfoil

- What happens farther from airfoil?
  - **waves** (shocks and expansions) **can interact**
  - expansion waves are carrying information from airfoil that the required turn angle has changed
  - so leading oblique shock gets weaker (curves)
    - shock angle decreases until it becomes a Mach wave
  - PM waves also curve as approaching $M\uparrow$
  - occurs again as waves from trailing edge propagate away from airfoil



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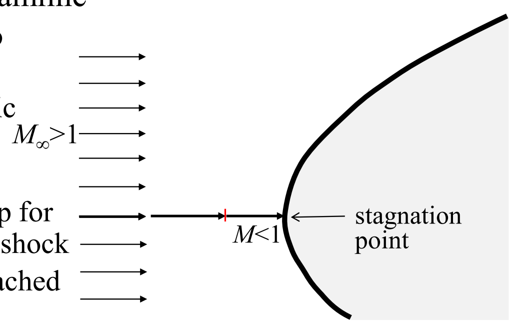
Blunt Body

- What happens for non-sharp (blunt) body in supersonic flow?
- Example symmetric body, zero angle of attack
- Consider center streamline
  - flow must come to rest at body
  - how can supersonic flow know to slow down?
    - 90° turn too sharp for attached oblique shock
    - can you have attached normal shock?



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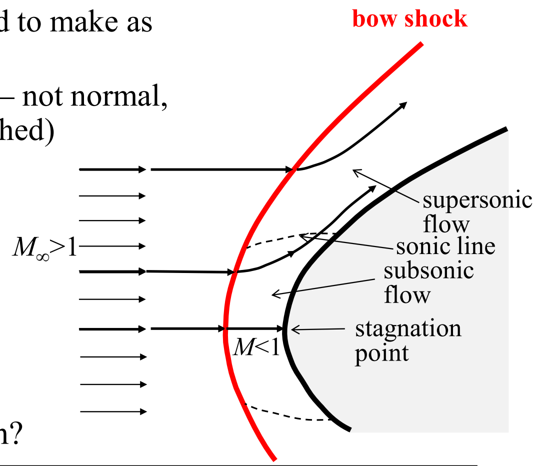
Blunt Body Example

- Nearby streamline
  - does flow need to make as sharp a turn?
  - weaker shock – not normal, oblique (detached)
- Rest of flowfield
  - progressively weaker shock required
  - how does shock know it can be weaker in supersonic region?



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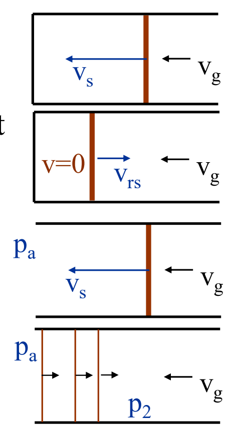
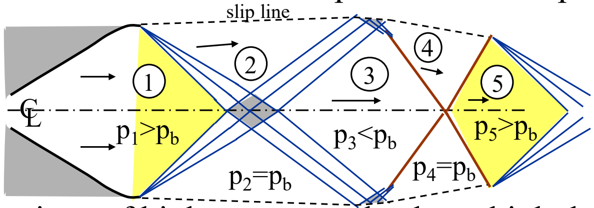
Reflected Waves

- What happens when wave "hits" a boundary ⇒ new BC
- Example: moving normal shock
  - if incident shock hits solid wall, get reflected (normal) shock - required to satisfy velocity BC (v=0)
  - if it hits open end, get reflected expansion waves - satisfy pressure BC ($p=p_a<p_2$)



Nozzle Exhaust: Reflected Waves

- CD nozzle under/over-expanded ⇒ BCs propagate



Underexpanded nozzle example

- Regions of high pressure, also have high density (and temperature); hot, dense gases emit light (radiation)

SR-71 at takeoff

from NASA