Overview

- The specific goals of this section are:
  1. Extend description for laminar, nonreacting jets to jet flames
  2. Introduce concept of conserved scalars
  3. Describe buoyancy effects on laminar jet flames
Laminar Jet Flames

- Reacting jet will be “similar” to nonreacting/mixing flow except
  1. Now have source and sink terms in the species conservation (and energy) equations
     - fuel and oxidizer will react to create products and convert chemical to thermal energy
  2. Must add diffusion of products (and intermediates/ radicals) to fuel and oxidizer diffusion
  3. Heat release will produce non-uniform temperature
     - $\rho = \text{constant}$ probably not a good assumption, will also lead to buoyancy effects (body force term)
     - $T$ dependent diffusivities will also vary with position

Jet Flames: Overall Fuel/Ox. Ratio

- Previously examined fuel jet exiting into infinite environment
  - important to recognize that is not always the case
- Based on total available fuel and oxidizer in system
  - overall excess oxidizer, overventilated (always the case for open air flame) $\dot{m}_F / \dot{m}_{O_F} < (\dot{m}_F / \dot{m}_{O_F})_{\text{stoich}}$
  - overall excess fuel, underventilated $\dot{m}_F / \dot{m}_{O_F} > (\dot{m}_F / \dot{m}_{O_F})_{\text{stoich}}$
Overventilated Jet Flame Description

- Fuel, oxidizer diffuse toward flame (reaction zone)
- Products (and heat) diffuse away
- Flame tip
  - final fuel burnout
  - location impacted by buoyancy
- Mixture fraction
  - continuous

Mixture Fraction

- Recall \( f \) def'n. (IX.5) \( f = \frac{\text{mass jet fluid @ point}}{\text{total mass @ point}} \)
- So what is the meaning of this definition when species originating in the jet are being consumed (e.g., to make products)
  - mixture fraction represents fraction of the mass at any point that originated in the jet
  - e.g., CO\(_2\) molecule’s C might originate from the fuel in the jet, while O’s came from the oxidizer
  - similarly N\(_2\) part of “products” for fuel jet exiting into air
Stoichiometric Mixture Fraction

- Assuming fuel is in the jet, $f_{\text{stoich}}$ represents fraction of product mass that originated from the fuel stream
  - mass conservation
    \[ v_F W_F = v_F W_F + v_{Ox} W_{Ox} \]

- Example methane jet exiting into air ($O_2$ and diluent)
  \[ CH_4 + 2(O_2 + 3.76N_2) \]  
  \[ f_{\text{stoich}} = \frac{16}{16 + 9.52(28.9)} = \frac{0.055}{1 + 2 \times 2 \times \frac{1}{0.233}} \]  

In terms of the actual fuel and $O_2$ in the two streams:

Conservation Equations

- Problem setup: rewrite conserv. eqns. IV.5,7,14,15 for simplifying assumptions
  1. same as nonreacting case: laminar, steady, $p$ const., axisymmetric, quiescent/infinite reservoir, Fickian binary diffusion, no axial diffusion (but $\rho \neq \text{const}$)
  2. plus for flame: vertical jet, normal thermal diffusion, no radiative transfer, negl. viscous dissipation, $Le=1$

- use similar normalizations to nonreacting jet

\[ x^* = \frac{x}{R} \quad u_x^* = \frac{u_x}{u_e} \quad \rho^* = \frac{\rho}{\rho_e} \]
Conservation Equations

• Mass

\[
\frac{\partial}{\partial x} \left( \rho u_x^* \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho u_r^* \right) = 0
\]

(IX.17)

• Axial momentum

\[
\left( \frac{\partial}{\partial x} \left( r \rho u_x^* u_x^* \right) \right) + \frac{\partial}{\partial r} \left( r \rho u_r^* u_x^* \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho u_x^* \right) = \frac{r}{Re} \frac{R}{u_e^*} \left( \rho_e^* - \rho_e^* \right)
\]

(IX.18)

\[
\text{axial conv.: radial conv.: radial diffusion: body force (buoyancy)}
\]

\[
\text{Re} = \frac{\rho u_e R}{\mu} \quad \mu \text{ local}
\]

– if \( Fr \ll \Delta \rho/\rho_e \Rightarrow \text{buoyancy controlled} \)

– if \( Fr \gg \Delta \rho/\rho_e \Rightarrow \text{buoyancy negligible} \)

Conservation Equations

• Species/Energy

– next we could write species conservation eq’ns. and energy equation in terms of \( T \)

– BUT run into problems with boundary conditions at flame/rxn zone = source/sink

– for example

\[
\frac{\partial}{\partial x} \left( r \rho u_x^* Y_i^* \right) + \frac{\partial}{\partial r} \left( r \rho u_r^* Y_i^* \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho u_x^* \right) = \frac{r^*}{m_e^*} \frac{m_i^*}{m_e^*}
\]

(IX.4)

axial conv.: radial conv.: radial diffusion: source/sink

• source/sink terms = 0 except in flame (rxn) zone, but where is the flame located?

• e.g., for flame sheet approx., locates itself to satisfy reactant mass flux ratio (IX.4)

\[
\frac{m_i^*}{m_f^*} = \frac{v_i}{v_f} \frac{W_i}{W_f}
\]
Conserved Scalars

• Solution to this problem - write equations in terms of **scalars that have no sources or sinks in the flow**
  
  – scalar that “exists” on both sides of flame and whose “integral” is constant (like $J_e$ and $m_e$ in nonreacting case)

• Examples
  
  – total enthalpy, $h_{sens} + h_{chem}$, if negligible: radiation, viscous dissipation, body force work (+ no conduction to bodies)
  
  – mixture fraction, $f$
  
  – elemental (atom) mass fraction $Z_i = \sum_{j=1}^{N} \mu_{ij} Y_j$
  
  $\mu_{ij}$ = mass proportion of element $i$ in species $j$
  
  $\mu_{H,CH} = 4/16$

Conserved Scalar Equations

• Species $\rightarrow$ Mixture fraction

$$\frac{\partial}{\partial x} \left( r^* \rho^* u^*_i f \right) + \frac{\partial}{\partial r} \left( r^* \rho^* u^*_i f \right) - \frac{1}{ScRe} \left( r^* \frac{1}{Re} \frac{\partial f}{\partial r} \right) = 0$$  \hspace{1cm} (IX.19)

• Energy $\rightarrow$ Total enthalpy

$$\frac{\partial}{\partial x} \left( r^* \rho^* u^* h^* \right) + \frac{\partial}{\partial r} \left( r^* \rho^* u^* h^* \right) - \frac{1}{ReSc} \left( r^* \frac{1}{ReSc} \frac{\partial h^*}{\partial r} \right) = 0$$  \hspace{1cm} (IX.20)

$$h^* \equiv \begin{cases} h - h_e \quad = 0 \text{ in pure ambient} \\ h_e - h_e \quad = 1 \text{ in pure jet fluid} \end{cases}$$  \hspace{1cm} (IX.21)

– same normalization as $f$
Boundary Conditions

\[ \eta = u^*_x, f, h^* \]

- General axisymmetric jet
  - \( @ \ r^* = 0, \text{ any } x^* \ \frac{\partial \eta}{\partial r} = 0, \ u_r^* = 0 \)
  - \( @ \ r^* = \infty, \quad ' ' \quad \eta = 0 \)
  - \( @ \ x^* = 0, \ r^* > 1 \quad \eta = 0 \)

- For top hat/uniform exit profile
  - \( @ \ x^* = 0, \ r^* \leq 1 \quad \eta = 1 \)

Conserv. Eqn./BC Summary

- Same boundary conditions on normalized variables
- Mom./species/energy eqns (IX.17-19) similar

\[ \left( r \ \frac{\partial}{\partial x} \right) = r^* \frac{R_g}{u^*_e} \left( \rho^*_u - \rho^* \right) \]

\[ \frac{\partial}{\partial x^*} \frac{1}{r^*} \left( r^* \ \frac{\partial}{\partial r^*} \left( r^* \ \frac{\partial}{\partial r^*} \frac{1}{r^*} \left( \frac{1}{r^*} \ \frac{\partial}{\partial r^*} \right) \right) = 0 \quad \text{species} \]

- no source/sink terms
  - \textit{except} in mom. if buoyancy effects not negligible \((Fr \ \text{small})\)
  - Schmidt and Lewis number dependence

- So if \(Le=1, h^*=f\); if \(Fr \ \text{large} \text{ and } Sc=1 \text{ then } u_x/u_e = f \)
- Still need to relate \(f, h^* \) to \(\rho, T \) \textit{unique relations if we assume fast chemistry (Da >>1)}
State Relationships

• Assuming ideal gases
  \[ \rho = \frac{pW}{RT} \]
• \( p=\text{const} \) a given; so to find \( \rho \), need \( T \) and \( \overline{W} \)
  - general process, starting with value for \( f \)
  \[ Y_i \quad \rightarrow \quad \frac{\sum Y_i/\overline{W}_i}{\sum Y_i c_{pi}} \quad \rightarrow \quad \overline{W} \quad \rightarrow \quad f \]
  e.g. for \( Le=1 \), \( h=J \)
  \[ h \quad \rightarrow \quad c_p \quad \rightarrow \quad T \]

So we need (state relations) \( f \rightarrow Y_i \)

\[ Y_i \] State Relationships

• For flame sheet approximation (\( v_F F + v_{Ox} Ox \rightarrow v_P P \)), linear relationships between \( f \) and \( Y_F, Y_{Ox}, Y_P \)
• Example of fuel jet exiting into air
  – for \( f > f_{\text{stoich}} \); on fuel side of flame
    • \( Y_{Ox} = 0 \)
    • \( Y_F (f=1)=1; Y_F (f_{\text{stoich}}) = 0 \)
    \[ \Rightarrow Y_F = (f - f_{\text{stoich}})/(1 - f_{\text{stoich}}) \]
    • \( Y_P = 1 - Y_F \)
    \[ \Rightarrow Y_P = (1 - f)/(1 - f_{\text{stoich}}) \]
  – similarly for \( f < f_{\text{stoich}} \)
    • \( Y_F = 0 \)
    • \( Y_P = f/f_{\text{stoich}} \)
    • \( Y_{Ox} = 1 - f/f_{\text{stoich}} \)

\[ f_{\text{stoich}} = \frac{W_P Y_F}{W_F Y_F} \]
State Relationships

- $h = ?$
  - for $Le=1$  
    
    \[
    h' = f \Rightarrow h = h_{Ox,\infty} + f(h_{F,e} - h_{Ox,\infty})
    \]
  
  Simple, linear relations for
  - flame sheet approx.
  - $Le_i=1$
  - $T$ if $c_p=constant$

- **Multistep chemistry** (flame sheet assumes single step)
  - for fast chemistry (i.e., chemical equilibrium) can still use state relation, but now fuel and oxidizer can coexist
  - broadens “flame”

History of Solution Approaches

- **Burke & Schumann (1928)**
  - “classical” analytic solution
    
    (e.g., see texts by Lewis & von Elbe; Kuo)
  - assumed $u_x=\text{const, } u_y=0$
    
    - so mom. equation not needed
      
      (can’t account for buoyancy)
  - assumed flame sheet, single $D$
  - reasonable results for $L_f$ of circular flames

- **Roper (1977)**
  - removed $u_x=\text{const. limitation}$
  - results improved for buoyant and non-circular jets

- **Numerical solutions** (e.g., Kee & Miller, *AIAA J.* 16, 1978)
  - can include finite rate kinetics, differential diffusion
Simplified Solutions

• Start with simplest case
• $Sc=1$ ($Le=1$), Constant Density, Const. “Properties”, Flame Sheet
  – $\rho$, $\mu=\rho v=\rho D=\rho \alpha = \text{constant}$
  – then all conservation equations become
    \[
    \frac{\partial}{\partial x} \left(r^* u^* \eta \right) + \frac{\partial}{\partial r^*} \left(r^* u^* \eta \right) - \frac{\partial}{\partial r^*} \left(r^* \frac{1}{Re} \frac{\partial \eta}{\partial r^*} \right) = 0
    \]
  – which gives identical solution as for nonreacting jet (IX.11,12)

Solutions (Constant Density)

• Flame Location
  – at $f=f_{\text{stoich}}$
  – from (IX.16)
    \[
    f_{\text{stoich}} = \frac{1}{1 + \left( \frac{O_2}{\text{fuel}} \right)_{\text{stoich}} \frac{Y_{\text{fuel},\infty}}{Y_{O_2,\infty}}}
    \]
  – if you dilute oxidizer $f_{\text{stoich}} \downarrow$ (flame gets bigger)
  – if you dilute fuel $f_{\text{stoich}} \uparrow$ (flame gets smaller)
  Discussion: why?
Solutions (Constant Density)

- **Flame Width**
  - \( r_{\text{flame}} \) where \( \xi = \xi(f_{\text{stoich}}) \)
  - from (IX.11)
    \[
    \xi_{\text{stoich}} = 2 \left( \frac{3}{8} \frac{Re_f R}{f_{\text{stoich}} x} \right)^{1/2} - 1
    \]
  - \( \xi \rightarrow r \) from (IX.10)
    \[
    \frac{r_{\text{flame}}}{x} = \frac{8}{\sqrt{3}} \frac{1}{Re_f} \left( 3 \frac{Re_f R}{8 f_{\text{stoich}} x} \right)^{1/2} - 1
    \]
  - consistent with previous flame size result
    \( r_{\text{flame}} \uparrow \) for \( f_{\text{stoich}} \downarrow \)

Discussion: why?

- **Flame Length**
  - flame tip reached when \( r_{\text{flame}} = 0 \)
    - get \( x_{\text{flame}} = L_f \) from (IX.22)
    \[
    L_f = \frac{3}{8} \frac{Re_f R}{f_{\text{stoich}}}
    \]
    \[
    = \frac{3}{8\pi} \frac{1}{f_{\text{stoich}}} \frac{Q_e}{v}
    \]
  - for given mass flowrate \( L_f \propto \frac{Q_e/D}{f_{\text{stoich}}} \) \( L_f \uparrow \) for 1) \( Q_e/D \uparrow \) or 2) \( f_{\text{stoich}} \downarrow \) (dilute ox.)

- reduce combustor length by having distributed nozzles
Solutions (Burke-Schumann)

- Earliest (approximate) solution
  - includes density variation
  - but assumes constant \( u_x \) and const. properties
    - but can get same result if one assumes \( \rho D=\text{const} \)
      (though actually \( \rho D \propto T^{0.5-0.8} \) for simple gases)

- Species equation (pseudo fuel = conserved scalar)
  \[
  \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho u_x Y_x^{'} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \rho u_y Y_y^{'} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( \rho D \frac{\partial Y_F^{'}}{\partial r} \right) = 0
  \]
  \[
  u_{x,ref} \frac{\partial}{\partial x} \left( Y_F^{'} \right) = D_{ref} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Y_F^{'}}{\partial r} \right)
  \]
  Solution by Bessel Functions (series solution)

Solutions (Variable Density)

- Fay extended Burke-Schumann approach
  - still \( Sc=Le=1 \) but with momentum equation, \( \rho \neq \text{const} \)
  - limited to \( Fr \) large (no buoyancy)

\[
L_f \propto L_f,\rho_{ref},\rho_{ref} = \text{const} \times \frac{\rho_F \rho_{ref}}{\rho_{ref}^2} \frac{1}{I_u(\rho_{ref}/\rho_f)}
\]

- \( \rho_F = \text{density in cold fuel} \)
- \( \rho_e = \text{density in cold oxidizer} \)
- \( \rho_f = \text{density at flame} \)
- \( I_u = \text{momentum integral (>1)} \)

\[
\frac{\rho_{ref}}{\rho_f} \approx \frac{\rho_F}{\rho_{ref}} \Rightarrow L_f > L_{f,\rho=\text{const}}
\]

- Roper
  - includes buoyancy (see Turns)

After Turns Table 9.2
Non-Analytic Solutions

- Four important issues not included in most analytic solutions ⇒ need for numerical solutions

1. Finite Rate Chemistry
   - can’t apply state relations, e.g., for $f \rightarrow Y_i$
   - solution will depend on chemical time scales and flow time scales (diffusion, residence times)

2. Differential Diffusion
   - analytic solutions assumed binary diffusion
   - e.g., product and intermediate species can diffuse at different rates

3. Axial Diffusion
   - can’t neglect in near-field or for low $u_e$ (e.g., low $Re_{jet}$)

4. Temperature dependent properties